Exercise 3.2-2

"Prove that $a^{\log_b c} = c^{\log_b a}$"

Let $\log_b a \triangleq k$, ie, $b^k = a$

$LHS = a^{\log_b c} = (b^k)^{\log_b c} = b^{\log_b (c^k)} = c^k = c^{\log_b a} = RHS.$

PROVED

PROBLEM

$\boxed{3-2 \ a, b, d, e}$

	A	B	
(a)	$\log^k n$	n^{ϵ}	$\log^k n = o(n^{\epsilon})$ [Y Y N N N]
(b)	n^k	c^n	$n^k = o(c^n)$ [Y Y N N N]
(d)	2^n	$2^{n/2}$	$2^n = \omega(2^{n/2})$ [N N Y Y N]
(e)	$n^{\log c}$	$c^{\log n}$	$n^{\log c} = o(c^{\log n})$ [Y Y N N N]

3.1 (c). Let $p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$ be a polynomial in n of degree k with $a_k > 0$. Prove that $p(n)$ is in $\Theta(n^k)$.

Solution:

I. $p(n) \in O(n^k)$:

$$\begin{aligned} p(n) &= a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0 \\ &= a_k \cdot n^k \cdot \left(1 + \frac{a_{k-1}}{a_k}\frac{1}{n} + \ldots + \frac{a_0}{a_k}\frac{1}{n^k}\right) \\ &\leq a_k \cdot n^k \cdot \left(1 + \frac{|a_{k-1}|}{a_k}\frac{1}{n} + \ldots + \frac{|a_0|}{a_k}\frac{1}{n^k}\right) \\ &\leq a_k \cdot n^k \cdot \left(1 + \frac{|a_{k-1}|}{a_k} + \ldots + \frac{|a_0|}{a_k}\right) \end{aligned}$$

for all $n \geq 1$. With

$$c := a_k \cdot \left(1 + \frac{|a_{k-1}|}{a_k} + \ldots + \frac{|a_0|}{a_k}\right)$$

$p(n) \leq c \cdot n^k$ for all $n \in \mathbb{N}$ holds. Thus $p(n) \in O(n^k)$.

II. $p(n) \in \Omega(n^k)$:

$$\begin{aligned} p(n) &= a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0 \\ &= a_k n^k \cdot \left(1 + \frac{a_{k-1}}{a_k}\frac{1}{n} + \ldots + \frac{a_0}{a_k}\frac{1}{n^k}\right) \end{aligned}$$

The expression within parentheses has limit 1 for $n \to \infty$. Thus, there is $n_0 \in \mathbb{N}$, such that this expression is $\geq 1/2$ for all $n \geq n_0$. Then, for $c := \frac{1}{2} a_k$ und $n \geq n_0$

$$p(n) \geq \frac{1}{2} a_k n^k = c \cdot n^k$$

holds. Thus $p(n) \in \Omega(n^k)$.

III. $p(n) \in \Theta(n^k)$ holds because of I. and II.

4-2 (a)

1. $T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\log n)$ {Master Method}

2. $T(n) = T(n/2) + \Theta(N) \Rightarrow T(n) = \Theta(n \log n)$ {Recursion Tree}

3. $T(n) = T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n)$ {M.M}

(b)

1. $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$ {MM}

2. $T(n) = 2T(n/2) + \Theta(N) \Rightarrow T(n) = \Theta(n^2)$ {Recursion Tree}

3. $T(n) = T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$ {MM}