

On the Soft Real-Time Optimality of Global EDF on Uniform Multiprocessors*

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Abstract

It has long been known that the global earliest-deadline-first (GEDF) scheduler is *soft real-time (SRT) optimal* for sporadic task systems executing on identical multiprocessor platforms. This notion of optimality requires deadline tardiness to be provably bounded for any task system that is feasible. In point of fact, GEDF is SRT optimal on identical platforms regardless of whether task execution is preemptive or non-preemptive. In recent years, there has been interest in extending these SRT optimality results to apply to uniform heterogeneous platforms in which processors may have different *speeds*. However, it was recently shown that non-preemptive GEDF is *not* SRT optimal on such platforms. The remaining case, preemptive GEDF, has turned out to be quite difficult to tackle and has remained open for a number of years. In this paper, this case is resolved by showing that preemptive GEDF is indeed SRT optimal on uniform platforms, provided a certain migration policy is used that ensures that tasks with earlier deadlines are scheduled on faster processors.

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1 Introduction

Multicore machines are capable of providing considerable processing power within a constrained size, weight, and power envelope. As a result, they are quickly emerging as the hardware platform of choice when hosting computationally intensive real-time applications. Although this emergence is well underway, research directed at finding good real-time schedulers for such platforms continues to be relevant. To fully harness the potential of multicore machines, such a scheduler should be reasonable to implement and preferably be *optimal*,

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meaning that it can correctly schedule any feasible task system. A task system is *feasible* if it is possible to schedule it (via *some* method) without causing timing-constraint violations.

One multiprocessor real-time scheduler that has received considerable attention is the preemptive *global earliest-deadline-first (GEDF)* scheduler.¹ GEDF generalizes the preemptive uniprocessor earliest-deadline-first (EDF) scheduler by using a single run queue to schedule all processors. EDF is optimal on uniprocessors and is straightforward to implement, so the motivation for considering its generalization in GEDF is clear. Unfortunately, in hard real-time (HRT) systems in which every deadline must be met, GEDF is not optimal, even for implicit-deadline sporadic task systems, due to the Dhall Effect [4]. Thus, from an optimality point of view, GEDF is in the same category as other “reasonable to implement” non-optimal schedulers, such as global fixed-priority schedulers and partitioned schedulers.

The situation is very different for soft real-time (SRT) systems, assuming timing correctness is defined by requiring deadline tardiness to be bounded. Under this notion of timing correctness, GEDF has long been known to be optimal for sporadic task systems, regardless of whether deadlines are implicit, constrained, or arbitrary [3, 9]. In fact, this optimality result extends to non-preemptive GEDF (NP-GEDF), under which jobs (*i.e.*, task invocations) are required to be non-preemptive. In contrast, global fixed-priority schedulers and partitioned schedulers (preemptive or non-preemptive) are not optimal in this SRT sense [2].

Heterogeneous multiprocessor platforms. The multicore revolution is currently undergoing a second wave of innovation in the form of heterogeneous processing elements. This evolution is further complicating software design processes that were already being challenged on account of the significant parallelism that exists in “conventional” (non-heterogeneous) multicore platforms. With heterogeneity, choices must be made when allocating hardware resources to software components. The need to resolve such choices can add considerable complexity to resource allocation.

Given the push towards greater heterogeneity, real-time schedulers for heterogeneous multiprocessor platforms are receiving considerable attention today. One heterogeneous platform model that has been studied extensively is the *uniform* model [5, 10]. This model differs from the conventional *identical* model in that processors may be of different *speeds*. The uniform model is viewed as an important stepping stone between the identical model and the *unrelated* model, which allows different tasks to experience unequal speed differences when moving from one processor to another. Additionally, the uniform model can be seen as an idealization of various *restricted processor supply* models. Under such a supply model, a processor may be partially available to a given application and hence can be approximated as executing at a slower speed.

GEDF on uniform platforms. Given the aforementioned SRT optimality results concerning GEDF and NP-GEDF on identical platforms, one might think that similar results can be easily shown for uniform platforms. However, we recently established the *non*-optimality of NP-GEDF in this context via a counterexample [13]. This counterexample pertains to a platform with only two processors. In the same paper, we were also able to establish the optimality of GEDF for such two-processor platforms. However, the general problem for the preemptive case has remained open. We ourselves have worked on this open problem intensively for over three years and have written about it in two papers [13, 15]. Others have considered it as well [11].

¹ References to “GEDF” without qualification should henceforth be taken to mean *preemptive* GEDF.

The elusiveness of this open problem is attributable to the fact that the feasibility condition for the uniform case is much more nuanced than that for the identical case. In particular, the uniform feasibility condition allows tasks to exist that have utilizations exceeding the speeds of certain processors, as explained in detail later. If such a task² τ_i is continuously scheduled on a processor that is “too slow,” then the tardiness of its jobs will increase without bound. In contrast, if τ_i were to be scheduled continuously on an identical platform, then its tardiness would not increase, assuming its utilization is at most one (which is required for feasibility). In particular, none of its jobs would be able to execute for longer than T_i time units, where T_i is its period. This very issue of tasks executing on processors that are “too slow” directly “breaks” the prior GEDF optimality proof for identical platforms [3] when extended to the uniform case.

Contributions. In this paper, we close the longstanding open problem mentioned above by proving that GEDF is SRT optimal on uniform platforms. Because the prior proof for identical platforms “breaks” in the uniform case, our result required an entirely new proof strategy. Together with the results in [15], the main result in this paper now completes the picture with respect to the SRT optimality of GEDF and NP-GEDF on uniform platforms. The version of GEDF that we consider uses an intuitive job migration policy: jobs with earlier deadlines are assigned to faster processors. Note that such policies are important in our context. For example, if a migration policy were to allow a task to continuously execute on a processor that is “too slow,” then as discussed above, tardiness would be unbounded.

The variant of GEDF that we consider is the first scheduler for uniform platforms to be proved optimal (HRT or SRT) in any general sense that avoids using the classical Level Algorithm [7] as a subroutine. The Level Algorithm is a very clever algorithm, but it preempts and migrates tasks at runtime frequently. The Level Algorithm was applied previously by Funk *et al.* [6] to obtain a scheduler that is HRT optimal for implicit-deadline periodic task systems. However, that scheduler is subject to excessive preemption and migration costs and was presented only as a byproduct in establishing feasibility conditions for implicit-deadline periodic task systems on uniform platforms (see (5) and (6) in Sec. 2). Additionally, we ourselves used the Level Algorithm in devising a semi-partitioned scheduler for implicit-deadline sporadic task systems that can be configured to be either HRT or SRT optimal on uniform platforms [14]. In that scheduler, the Level Algorithm can be configured to preempt and migrate tasks less frequently, at the expense of greater tardiness bounds.

Organization. The rest of this paper is organized as follows. In Sec. 2, we provide necessary technical definitions and background. Then, in Sec. 3, we establish tardiness bounds for periodic tasks, assuming all jobs execute for their exact worst-case cost. Next, in Sec. 4, we explain how to extend our results to apply to sporadic tasks where jobs are not assumed to always execute for their worst-case cost. Finally, we conclude in Sec. 5.

2 Background

When considering the issue of heterogeneity, processor speed becomes an important issue. The *speed* of a processor refers to the amount of work completed in one time unit when a task is executed on that processor. The following speed-oriented classification of multiprocessor platforms has been widely accepted [5, 10].

² The task models considered herein are formally defined in Sec. 2.

- **Identical multiprocessors.** Every task is executed on any processor at the same speed, which is usually normalized to be 1.0 for simplicity.
- **Uniform multiprocessors.** Different processors may have different speeds, but on a given processor, every task is executed at the same speed. The speed of processor p is denoted s_p .
- **Unrelated multiprocessors.** The execution speed of a task depends on both the processor on which it is executed and the task itself, *i.e.*, a given processor may execute different tasks at different speeds.

In this paper, our focus is uniform multiprocessors. Specifically, we consider the scheduling of a set τ of n sequential tasks on a uniform platform³ π consisting of m processors, where the processors are indexed by their speeds in non-increasing order, *i.e.*, $s_i \geq s_{i+1}$ for $i = 1, 2, \dots, m-1$. We denote the sum of k largest speeds on π as $S_k = \sum_{i=1}^k s_i$ for $k = 1, 2, \dots, m$. Furthermore, we assume $m \geq 2$, for otherwise, uniprocessor analysis can be applied. We also assume $n \geq m$, for otherwise, there is no point in ever scheduling any task on any of the $m-n$ slower processors, so m and n can conceptually be deemed as equal in this case.

We consider tasks that are either *sporadic* or *periodic*. A sporadic (respectively, periodic) task τ_i releases a sequence of *jobs* with a minimum (respectively, exact) separation of T_i time units between invocations. The parameter T_i is called the *period* of τ_i . τ_i also has a *worst-case execution requirement* C_i , which is defined as the maximum execution time of any one job (invocation) of τ_i on a unit-speed processor. We let $C_{\max} = \max\{C_i \mid 1 \leq i \leq n\}$. The *utilization* of task τ_i is given by $u_i = C_i/T_i$. We assume that tasks are indexed in non-increasing order by utilization, *i.e.*, $u_i \geq u_{i+1}$ for $i = 1, 2, \dots, n-1$. We denote the sum of k largest utilizations in τ as $U_k = \sum_{i=1}^k u_i$ for $k = 1, 2, \dots, n$. Furthermore, we denote the ratio between the largest and the smallest utilizations as $\rho = u_1/u_n$. As for scheduling, we assume that deadlines are *implicit*, *i.e.*, each task τ_i has a *relative deadline* parameter equal to its period T_i . Furthermore, because tasks are *sequential*, intra-task parallelism is not allowed, and an invocation of a task cannot commence execution until all previous invocations of that task complete. Throughout the paper, we assume that time is continuous.

The j^{th} job (or invocation) of task τ_i is denoted as $J_{i,j}$. Job $J_{i,j}$ has a release (or arrival) time denoted $a_{i,j}$, an absolute deadline denoted $d_{i,j} = a_{i,j} + T_i$, and a completion (or finish) time denoted $f_{i,j}$. The *tardiness* of job $J_{i,j}$ is defined by $\max\{0, f_{i,j} - d_{i,j}\}$ and its *response time* by $f_{i,j} - a_{i,j}$. The *tardiness* of task τ_i in some schedule is the maximum tardiness of any of its jobs in that schedule. A job is *pending* if it is released but has not completed, and is *ready* if it is pending and all preceding jobs of the same task have completed.

A task set is *HRT schedulable* (respectively, *SRT schedulable*) under a given scheduler if each task can be guaranteed zero (respectively, bounded) tardiness under that scheduler. A task set is *HRT feasible* (respectively, *SRT feasible*) if it is HRT schedulable (respectively, SRT schedulable) under some scheduler. A given scheduler is *HRT optimal* (respectively, *SRT optimal*) if any HRT feasible (respectively, SRT feasible) task set is HRT schedulable (respectively, SRT schedulable) under it.

Ideal schedule. We define an *ideal multiprocessor* $\pi_{\mathcal{I}}$ for the task set τ as one that consists of n uniform processors where the speeds of the n processors exactly match the utilizations of the n tasks in τ , respectively, *i.e.*, the speed of the i^{th} processor is $s_i^{\mathcal{I}} = u_i$ for $i = 1, 2, \dots, n$. We define the *ideal schedule* \mathcal{I} to be the partitioned schedule for τ on $\pi_{\mathcal{I}}$, where each task

³ We use the terminology “uniform platform” and “uniform multiprocessor” interchangeably.

τ_i in τ is assigned to the processor of speed $s_i^{\mathcal{I}}$. Then, in \mathcal{I} , every job in τ commences execution at its release time and completes execution within one period (it exactly executes for one period if and only if its actual execution requirement matches its worst-case execution requirement). Thus, all deadlines are met in \mathcal{I} .

Definition of lag. Let $A(\mathcal{S}, \tau_i, t_1, t_2)$ denote the cumulative processor capacity allocated to task τ_i in an arbitrary schedule \mathcal{S} within the time interval $[t_1, t_2]$. By the definition of the ideal schedule \mathcal{I} ,

$$A(\mathcal{I}, \tau_i, t_1, t_2) \leq u_i \cdot (t_2 - t_1). \quad (1)$$

Also, if τ_i releases jobs periodically and every job's actual execution requirement equals its worst case of C_i , then for any t_1 and t_2 such that $a_{i,1} \leq t_1 \leq t_2$,

$$A(\mathcal{I}, \tau_i, t_1, t_2) = u_i \cdot (t_2 - t_1). \quad (2)$$

For an arbitrary schedule \mathcal{S} , we denote the difference between the allocation to a task τ_i in \mathcal{I} and in \mathcal{S} within time interval $[0, t]$ as

$$\text{lag}(\tau_i, t, \mathcal{S}) = A(\mathcal{I}, \tau_i, 0, t) - A(\mathcal{S}, \tau_i, 0, t). \quad (3)$$

The lag function captures the allocation difference between an arbitrary actual schedule \mathcal{S} and the ideal schedule \mathcal{I} . If $\text{lag}(\tau_i, t, \mathcal{S})$ is positive, then \mathcal{S} has performed less work on τ_i until time t , i.e., τ_i is “under-allocated,” while if $\text{lag}(\tau_i, t, \mathcal{S})$ is negative, then τ_i is “over-allocated.” Also, for any two time instants t_1 and t_2 such that $t_1 \leq t_2$, we have

$$\text{lag}(\tau_i, t_2, \mathcal{S}) = \text{lag}(\tau_i, t_1, \mathcal{S}) + A(\mathcal{I}, \tau_i, t_1, t_2) - A(\mathcal{S}, \tau_i, t_1, t_2). \quad (4)$$

A necessary and sufficient SRT feasibility condition. For HRT task sets, Funk *et al.* [6] showed that a set of implicit-deadline periodic tasks is feasible on a uniform platform if and only if the following constraints hold:

$$U_n \leq S_m, \quad (5)$$

$$U_k \leq S_k, \text{ for } k = 1, 2, \dots, m-1. \quad (6)$$

It can be shown that this constraint set is also a feasibility condition for implicit-deadline *sporadic* tasks. Furthermore, the sufficiency of this constraint set for HRT task sets implies its sufficiency for SRT task sets. In fact, these constraints are necessary for SRT task sets as well. To see this, note that if $U_n > S_m$ holds (contrary to (5)), then the total workload over-utilizes the platform, so some task will be unboundedly tardy if tasks release jobs as soon as possible and always execute for their worst-case costs. Furthermore, if $U_k > S_k$ holds (contrary to (6)), then the set of k highest-utilization tasks will be “under-allocated” at every time instant if they release jobs as soon as possible and always execute for their worst-case costs. This is because k tasks can be allocated to at most k processors at any time instant and the sum of the speeds of *any* k processors can be at most S_k . Thus, $\sum_{i=1}^k \text{lag}(\tau_i, t, \mathcal{S})$ will increase unboundedly, which implies that $\text{lag}(\tau_i, t, \mathcal{S})$ will increase unboundedly for some i . This implies that task τ_i will be unbounded tardy.

To summarize, (5) and (6) are also a necessary and sufficient feasibility condition for SRT task sets. Therefore, when henceforth referring to this constraint set as a feasibility condition, we do not need to further specify whether this is meant for HRT or SRT task sets.

The GEDF scheduler. From a scheduling point of view, uniform platforms differ from identical ones in a significant way: on a uniform platform, besides *which* tasks are scheduled at any time, the scheduler must also decide *where* they are scheduled, because different processors may have different speeds. Thus, we must refine the notion of GEDF scheduling to be clear about where tasks are scheduled. In this paper, we assume the following.

(G) If at most m jobs are ready, then all ready jobs are scheduled; otherwise, the m ready jobs with earliest deadlines are scheduled. At any time, the ready job with the k^{th} earliest deadline is scheduled on the k^{th} fastest processor for any k . (Note that this implies that a job may migrate from one processor to another during its execution.) Deadline ties are broken arbitrarily.

Henceforth, all references to GEDF are assumed to mean GEDF as defined by Policy (G), unless specified otherwise.

At a given time instant t , we say that a *task* is *pending* if it has any pending jobs at time t . If task τ_i is pending at time t , then it has exactly one ready job $J_{i,j}$ at time t . The deadline of that job is called the *effective* deadline of τ_i at time t and is denoted $d_i(t) = d_{i,j}$. Similarly, the *effective* release time of τ_i at time t is denoted $a_i(t) = a_{i,j}$. Because deadlines are implicit, $d_i(t) = a_i(t) + T_i$. Also, $a_i(t) \leq t$ holds, for otherwise, $J_{i,j}$ would not be ready at time t . The following lemma gives a sufficient lag-based condition for a task to be pending.

► **Lemma 1.** *If $\text{lag}(\tau_i, t, \mathcal{S}) > 0$, then τ_i is pending at time t in \mathcal{S} .*

Proof. Suppose that $\text{lag}(\tau_i, t, \mathcal{S}) > 0$ holds but τ_i is not a pending task at time t in \mathcal{S} . Then, all jobs of τ_i released at or before t have completed by time t . Thus, letting W denote the total actual execution requirement of all such jobs, we have $A(\mathcal{S}, \tau_i, 0, t) = W$. In the ideal schedule \mathcal{I} , only released jobs can be scheduled and will not execute for more than their actual execution requirement. Thus, $A(\mathcal{I}, \tau_i, 0, t) \leq W$ holds as well. By (3), these facts imply $\text{lag}(\tau_i, t, \mathcal{S}) = A(\mathcal{I}, \tau_i, 0, t) - A(\mathcal{S}, \tau_i, 0, t) \leq 0$. This contradicts our assumption that $\text{lag}(\tau_i, t, \mathcal{S}) > 0$ holds. ◀

Prior results and the remaining open problem. As mentioned earlier in Sec. 1, the key difficulty faced when trying to extend prior tardiness analysis for identical platforms [3, 9] to uniform ones is that, in the uniform case, tasks can execute on processors that are “too slow.” The specific problematic property required in the prior analysis is the following.

(P) If any job $J_{i,j}$ executes continuously, then it must complete within T_i time units, regardless of the processor on which it executes.

Clearly, (P) can be violated if $J_{i,j}$ executes entirely on processors of speed less than u_i .

It is tempting to obviate all problematic issues pertaining to Property (P) by simply enforcing scheduling policies that uphold it. Such an approach was taken by Tong and Liu [11], who considered a variant of GEDF in which each task τ_i is only allowed to execute on processors with speed at least u_i . However, such a requirement results in non-optimal scheduling. For example, Tong and Liu’s GEDF variant is not able to correctly schedule a set of two tasks on two processors such that $u_1 = 2$, $u_2 = 2$, $s_1 = 3$, and $s_2 = 1$. From (5) and (6), we see that this task set is feasible. However, under their algorithm, a task τ_i can only execute on a processor of speed at least u_i , so both tasks in this example must exclusively execute on the processor with speed $s_1 = 3$. That processor will be over-utilized if each task releases jobs as soon as possible and always executes for its worst-case cost, since $u_1 + u_2 = 4 > s_1 = 3$.

In other work, we successfully eliminated the need for Property (P) by relaxing the task model to allow consecutive jobs of the same task to execute in parallel. Under this relaxed task model, we were able to establish the SRT optimality of GEDF on uniform platforms [12].

For the sequential task model being considered in this paper, we also found that Property (P) is not necessary if the underlying uniform platform has only two processors [13]. However, we believe that it is unlikely that the particular proof strategy used in that paper can be extended to the more general m -processor case.

To clearly place our contribution in its proper context in light of this prior work, we emphasize here several assumptions made hereafter in solving the open problem considered in this paper:

- we are interested in *any* feasible task set, *i.e.*, no constraints on task utilizations other than (5) and (6) are assumed;
- intra-task parallelism is strictly forbidden, *i.e.*, jobs of the same task must execute in sequence;
- the uniform platform may have m processors, where m can be any positive integer value.

3 Periodic Tasks with an Assumption

In this section, we prove tardiness bounds for an arbitrary feasible *periodic* task set on a uniform platform π , under the following assumption.

(A) Every job of any task executes for its worst-case execution requirement of C_i .

In the rest of this section, we restrict attention to a periodic task set τ for which Assumption (A) holds. Our objective is to derive tardiness bounds when the GEDF scheduler is used to schedule τ . We do so by reasoning about lag values in an arbitrary GEDF schedule \mathcal{S} for τ . The concept of lag is useful for our purposes because a task that has positive lag at one of its deadlines will have a tardy job. Focusing on periodic task sets where Assumption (A) holds facilitates much of the lag-based reasoning that is needed. Later, in Sec. 4, we will explain how to extend our results so that Assumption (A) is not required and so that sporadic tasks can be supported.

Properties of lag values and deadlines. We begin by proving a number of properties concerning lag values and deadlines and relationships between the two. The first such property is given in the following lemma.

► **Lemma 2.** *If task τ_i is pending at time t in \mathcal{S} , then its effective deadline $d_i(t)$ has the following relationship with $\text{lag}(\tau_i, t, \mathcal{S})$.*

$$t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} < d_i(t) \leq t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} + T_i \quad (7)$$

Proof. Let $e_i(t)$ denote the remaining execution requirement for the ready job $J_{i,j}$ of τ_i at time t in \mathcal{S} . Because $J_{i,j}$ is ready at time t , it has not finished execution by then, so

$$0 < e_i(t) \leq C_i. \quad (8)$$

Furthermore, all jobs of τ_i prior to $J_{i,j}$ have completed by time t in \mathcal{S} . Let W denote the total execution requirement for all of these jobs. Then, given Assumption (A),⁴

$$A(\mathcal{S}, \tau_i, 0, t) = W + C_i - e_i(t). \quad (9)$$

⁴ Without Assumption (A), $J_{i,j}$ may execute in total for less than its worst-case execution requirement of C_i , and therefore only “ \leq ” can be claimed in (9).

Now consider the ideal schedule \mathcal{I} . In it, all jobs of τ_i prior to $J_{i,j}$ have complete by time $a_i(t) \leq t$. Given Assumption (A), within $[a_i(t), t]$, \mathcal{I} continuously⁵ executes job $J_{i,j}$ at a rate of u_i . Thus,

$$\mathsf{A}(\mathcal{I}, \tau_i, 0, t) = W + (t - a_i(t))u_i. \quad (10)$$

Therefore, an expression for $\text{lag}(\tau_i, t, \mathcal{S})$ can be derived as follows.

$$\begin{aligned} \text{lag}(\tau_i, t, \mathcal{S}) &= \{\text{by (3)}\} \\ &\quad \mathsf{A}(\mathcal{I}, \tau_i, 0, t) - \mathsf{A}(\mathcal{S}, \tau_i, 0, t) \\ &= \{\text{by (9) and (10)}\} \\ &\quad (t - a_i(t))u_i - (C_i - e_i(t)) \\ &= \{\text{because } d_i(t) = a_i(t) + T_i\} \\ &\quad (t - d_i(t) + T_i)u_i - (C_i - e_i(t)) \\ &= \{\text{because } T_i \cdot u_i = C_i\} \\ &\quad (t - d_i(t))u_i + e_i(t) \end{aligned}$$

By (8) and the above expression for $\text{lag}(\tau_i, t, \mathcal{S})$, we have

$$(t - d_i(t))u_i < \text{lag}(\tau_i, t, \mathcal{S}) \leq (t - d_i(t))u_i + C_i. \quad (11)$$

Rearranging the terms in (11) yields (7). \blacktriangleleft

► **Corollary 3.** *If $\text{lag}(\tau_i, t, \mathcal{S}) \leq L$ for all t , then the tardiness of task τ_i is at most L/u_i .*

Proof. Suppose that

$$\text{lag}(\tau_i, t, \mathcal{S}) \leq L \quad (12)$$

holds but τ_i has tardiness exceeding L/u_i . Then, there exists a job $J_{i,j}$ that is still pending at some time $t \geq d_{i,j}$ where

$$t - d_{i,j} > L/u_i. \quad (13)$$

Because $J_{i,j}$ is pending at time t , τ_i is a pending task at time t and its ready job at time t cannot be a job released later than $J_{i,j}$. Thus, τ_i 's effective deadline at t satisfies $d_i(t) \leq d_{i,j}$. Therefore,

$$\begin{aligned} t - d_i(t) &\geq t - d_{i,j} \\ &> \{\text{by (12)}\} \\ &\quad L/u_i \\ &\geq \{\text{by (13)}\} \\ &\quad \text{lag}(\tau_i, t, \mathcal{S})/u_i. \end{aligned}$$

That is, $t - \text{lag}(\tau_i, t, \mathcal{S})/u_i > d_i(t)$, which contradicts Lemma 2. \blacktriangleleft

Recall that if $\text{lag}(\tau_i, t, \mathcal{S})$ is negative, then τ_i is over-allocated in schedule \mathcal{S} compared to schedule \mathcal{I} . However, the actual schedule \mathcal{S} cannot execute jobs that are not released and therefore can never get more than a full job “ahead” of \mathcal{I} . Thus, we have the following trivial lower bound⁶ on $\text{lag}(\tau_i, t, \mathcal{S})$, which we state without proof.

⁵ Without Assumption (A), $J_{i,j}$ might not execute “continuously,” so only “ \leq ” can be claimed in (10).

⁶ A tighter bound is possible, but this simple bound is sufficient for our purposes.

► **Lemma 4.** $\text{lag}(\tau_i, t, \mathcal{S}) \geq -C_{\max}$.

The following lemma uses the relationship between effective deadlines and lag values established in Lemma 2 to obtain a sufficient lag-based condition for one task to have an earlier effective deadline than another.

► **Lemma 5.** *If tasks τ_i and τ_k are both pending at time t , and if*

$$\text{lag}(\tau_i, t, \mathcal{S}) \geq \frac{u_i}{u_k} \cdot \text{lag}(\tau_k, t, \mathcal{S}) + C_i \quad (14)$$

holds, then $d_i(t) < d_k(t)$.

Proof.

$$\begin{aligned} d_i(t) &\leq \{\text{by Lemma 2}\} \\ &\quad t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} + T_i \\ &\leq \{\text{by (14)}\} \\ &\quad t - \frac{\frac{u_i}{u_k} \cdot \text{lag}(\tau_k, t, \mathcal{S}) + C_i}{u_i} + T_i \\ &= \{\text{canceling } u_i \text{ and using } C_i/u_i = T_i\} \\ &\quad t - \frac{\text{lag}(\tau_k, t, \mathcal{S})}{u_k} \\ &< \{\text{by Lemma 2}\} \\ &\quad d_k(t) \end{aligned}$$

The lemma follows. ◀

► **Corollary 6.** *If tasks τ_i and τ_k are both pending at time t and if $\text{lag}(\tau_i, t, \mathcal{S}) \geq \rho \cdot \text{lag}(\tau_k, t, \mathcal{S}) + C_{\max}$ holds, where $\rho = u_1/u_n$, then $d_i(t) < d_k(t)$.*

Proof. Because tasks are indexed from highest utilization to lowest, $\text{lag}(\tau_i, t, \mathcal{S}) \geq \rho \cdot \text{lag}(\tau_k, t, \mathcal{S}) + C_{\max} = \frac{u_1}{u_n} \cdot \text{lag}(\tau_k, t, \mathcal{S}) + C_{\max} \geq \frac{u_i}{u_k} \cdot \text{lag}(\tau_k, t, \mathcal{S}) + C_i$. By Lemma 5, the corollary follows. ◀

Proof strategies for deriving tardiness bounds. Given the relationships established above between lag values and deadlines, we are now ready to derive tardiness bounds. According to Corollary 3, lag bounds directly imply tardiness bounds. Thus, one natural strategy is to attempt to derive n individual lag bounds, one per task. However, we were unable to make this strategy work. Intuitively, this is because, in deriving n individual per-task lag bounds, we must consider how all tasks interact as they are scheduled together. When doing this, it is difficult to avoid a case explosion that causes the entire proof to collapse. In particular, the feasibility condition given by (5) and (6) must ultimately be exploited in the proof. Every attempt we made in deriving per-task lag bounds resulted in a case explosion that was so unwieldy, we could not discern how (5) and (6) could possibly factor into the proof.

Notice that (5) simply requires that the platform is not over-utilized, which is something required in reasoning about identical platforms as well. The constraints in (6), however, are unique to the uniform case. Observe that these constraints reference the sum of the k largest utilizations and speeds. Accordingly, we switched from working on proof strategies that focus on per-task lag bounds to one that focuses on the sum of the k largest lag values.

Our proof strategy, formally explained. In order to describe this proof strategy more formally, we let $\hat{\tau}_\ell(t)$ denote the task that has the ℓ^{th} largest lag at time instant t , with ties broken arbitrarily. We also denote the ℓ^{th} largest lag at time instant t as $\text{Lag}_\ell(t)$, i.e., $\text{Lag}_\ell(t) = \text{lag}(\hat{\tau}_\ell(t), t, \mathcal{S})$. Furthermore, we let $\mathcal{T}_\ell(t)$ denote the set of tasks corresponding to $\text{Lag}_1(t), \text{Lag}_2(t), \dots, \text{Lag}_\ell(t)$, i.e., $\mathcal{T}_\ell(t) = \{\hat{\tau}_1(t), \hat{\tau}_2(t), \dots, \hat{\tau}_\ell(t)\}$.

To derive tardiness bounds, we show that the following $m+1$ inequalities, $(B_1), \dots, (B_m)$, and (B_n) , hold at any time t .

Inequality Set (B):

$$\text{Lag}_1(t) \leq \beta_1 \quad (B_1)$$

$$\text{Lag}_1(t) + \text{Lag}_2(t) \leq \beta_2 \quad (B_2)$$

$$\text{Lag}_1(t) + \text{Lag}_2(t) + \text{Lag}_3(t) \leq \beta_3 \quad (B_3)$$

$$\vdots \quad \vdots$$

$$\text{Lag}_1(t) + \text{Lag}_2(t) + \dots + \text{Lag}_k(t) \leq \beta_k \quad (B_k)$$

$$\vdots \quad \vdots$$

$$\text{Lag}_1(t) + \text{Lag}_2(t) + \dots + \text{Lag}_m(t) \leq \beta_m \quad (B_m)$$

$$\text{Lag}_1(t) + \text{Lag}_2(t) + \dots + \text{Lag}_n(t) \leq \beta_n \quad (B_n)$$

If all constraints in the inequality set (B) hold at all time instants t , where $\beta_1, \beta_2, \dots, \beta_m$, and β_n are constants (which will depend on task-set parameters), then, by the definition of $\text{Lag}_1(t)$, β_1 is an upper bound on $\text{lag}(\tau_i, t, \mathcal{S})$ for any i and for any t . Given such an upper bound, by Corollary 3, tardiness bounds will follow.

In order to prove that the constraints in (B) hold at all time instants t , we must carefully define $\beta_1, \beta_2, \dots, \beta_m$, and β_n . They are defined as follows.

$$\beta_1 = x_1 \quad (X_1)$$

$$\beta_2 = \beta_1 + x_2 \quad (X_2)$$

$$\beta_3 = \beta_2 + x_3 \quad (X_3)$$

$$\vdots \quad \vdots$$

$$\beta_k = \beta_{k-1} + x_k \quad (X_k)$$

$$\vdots \quad \vdots$$

$$\beta_m = \beta_{m-1} + x_m \quad (X_m)$$

$$\beta_n = \beta_m + x_n \quad (X_n)$$

where

$$x_n = -(n - m - 1) \cdot C_{\max} \quad (Y_1)$$

$$x_m = (n - m + 1) \cdot C_{\max} \quad (Y_2)$$

$$x_i = \rho \cdot x_{i+1} + C_{\max}, \quad \text{for } i = m - 1, m - 2, \dots, 1 \quad (Y_3)$$

Note that, in (Y₃), $\rho = u_1/u_n$.

To see that $\beta_1, \beta_2, \dots, \beta_m$, and β_n are well-defined, observe that x_n and x_m can be directly calculated by (Y₁) and (Y₂) for any given task set. Then, $x_{m-1}, x_{m-2}, \dots, x_1$ can be calculated inductively by (Y₃). Finally, $\beta_1, \beta_2, \dots, \beta_m$, and β_n can be calculated by (X₁), ..., (X_m), and (X_n).

Formal derivation of tardiness bounds. Having set up our proof strategy, we next present a critical mathematical property of the lag and Lag functions when they are viewed as a function of t .

► **Property 7.** *For a given task τ_i and a given schedule \mathcal{S} , $\text{lag}(\tau_i, t, \mathcal{S})$ is a continuous function of t . For a given schedule \mathcal{S} , $\text{Lag}_\ell(t)$ is a continuous function of t for each ℓ .*

Proof. $\text{lag}(\tau_i, t, \mathcal{S})$ is a continuous function of t because, by (3), $\text{lag}(\tau_i, t, \mathcal{S}) = A(\mathcal{I}, \tau_i, 0, t) - A(\mathcal{S}, \tau_i, 0, t)$, and $A(\mathcal{I}, \tau_i, 0, t)$ and $A(\mathcal{S}, \tau_i, 0, t)$ are both (clearly) continuous functions of t . Furthermore, since taking the maximum value of a set of continuous functions is also a continuous function, $\text{Lag}_1(t)$ is a continuous function of t . For similar reasons, $\text{Lag}_2(t)$, $\text{Lag}_3(t), \dots, \text{Lag}_n(t)$ are all continuous functions of t as well. ◀

We are now ready to prove our main theorem.

► **Theorem 8.** *At every time instant $t \geq 0$, each inequality in the set (B) holds.*

Proof. Suppose, to the contrary, that the statement of the theorem is not true, and let t_c denote the first time instant such that *any* inequality in (B) is false. We show that the existence of t_c leads to a contradiction.

► **Claim 9.** $t_c > 0$.

Proof. It follows by induction using (Y₂) and (Y₃) (and our assumption from Sec. 2 that $n \geq m$) that $x_i > 0$ for $i = 1, 2, \dots, m$. By induction again, this time using (X₁), ..., (X_m), it further follows that $\beta_i > 0$ for $i = 1, 2, \dots, m$. Finally, by (X_m), (X_n), (Y₁), and (Y₂), $\beta_n = \beta_{m-1} + 2C_{\max} > 0$. Thus, because $\text{Lag}_i(0) = 0$ holds for all i , all of the inequalities in (B) are true at time 0, implying that $t_c > 0$. ◀

Let $t_c^- = t_c - \varepsilon$, where $\varepsilon \rightarrow 0^+$.⁷ By Claim 9, $t_c^- \geq 0$, i.e., t_c^- is well-defined. Because t_c is the *first* time instant at which *any* inequality in (B) is false, *all* such inequalities hold prior to t_c , including at time t_c^- . Also, because the length of the interval $[t_c^-, t_c]$ is arbitrarily small, a task scheduled on a processor at time t_c^- will be continuously scheduled within $[t_c^-, t_c]$.

We call an inequality in (B) *critical* if and only if it is false at time t_c . If (B_k) is critical, then $\text{Lag}_1(t_c^-) + \dots + \text{Lag}_k(t_c^-) = \beta_k$. This is because (B_k) holds for any time instant before t_c but is falsified at t_c and the left-hand-side of (B_k) is a continuous function of t , by Property 7.⁸ We now consider two cases, which depend on which inequalities are critical.

Case 1: (B_n) is critical. In this case,

$$\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \dots + \text{Lag}_n(t_c^-) = \beta_n. \quad (15)$$

Therefore,

⁷ ε does not have to be infinitely close to 0. Instead, it only needs to be a sufficiently small positive constant. However, the criteria for “sufficiently small” are rather tedious, so we merely define $\varepsilon \rightarrow 0^+$ here for simplicity. Whenever ε is used, we will further elaborate on its definition in that context.

⁸ If $\text{Lag}_1(t_c^-) + \dots + \text{Lag}_k(t_c^-) < \beta_k$, then a time $t \in [t_c^-, t_c]$ must exist such that $\text{Lag}_1(t) + \dots + \text{Lag}_k(t) = \beta_k$. Therefore, a smaller ε could have been selected so that $t_c^- = t$.

$$\begin{aligned}
 & \text{Lag}_m(t_c^-) + \text{Lag}_{m+1}(t_c^-) + \cdots + \text{Lag}_n(t_c^-) \\
 &= \{\text{by (15)}\} \\
 &\quad \beta_n - (\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_{m-1}(t_c^-)) \\
 &\geq \{\text{because } (B_{m-1}) \text{ holds at time } t_c^-\} \\
 &\quad \beta_n - \beta_{m-1} \\
 &= \{\text{by (X}_m), (\text{X}_n), (\text{Y}_1), \text{ and (Y}_2)\} \\
 &\quad 2C_{\max}. \tag{16}
 \end{aligned}$$

Furthermore, by definition, $\text{Lag}_m(t_c^-) \geq \text{Lag}_{m+1}(t_c^-) \geq \cdots \geq \text{Lag}_n(t_c^-)$, so $\text{Lag}_m(t_c^-)$ is at least the average of these $n - m + 1$ values. Therefore,

$$\begin{aligned}
 \text{Lag}_m(t_c^-) &\geq \frac{\text{Lag}_m(t_c^-) + \text{Lag}_{m+1}(t_c^-) + \cdots + \text{Lag}_n(t_c^-)}{n - m + 1} \\
 &\geq \{\text{by (16)}\} \\
 &\quad \frac{2C_{\max}}{n - m + 1} \\
 &> \{\text{because } C_{\max} > 0 \text{ and } n \geq m\} \\
 &\quad 0. \tag{17}
 \end{aligned}$$

Because $\text{Lag}_m(t_c^-)$ denotes the m^{th} largest lag at time t_c^- , (17) implies that, at time t_c^- , at least m tasks have positive lag. Thus, by Lemma 1, at least m tasks pending at time t_c^- . Therefore, all of the m processors are busy during the time interval $[t_c^-, t_c]$. Thus, by (5), the total lag in the system does not increase during the interval $[t_c^-, t_c]$. That is, $\text{Lag}_1(t_c) + \text{Lag}_2(t_c) + \cdots + \text{Lag}_n(t_c) = \sum_{i=1}^n \text{lag}(\tau_i, t_c, \mathcal{S}) \leq \sum_{i=1}^n \text{lag}(\tau_i, t_c^-, \mathcal{S}) = \text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_n(t_c^-)$, which by (15), implies

$$\text{Lag}_1(t_c) + \text{Lag}_2(t_c) + \cdots + \text{Lag}_n(t_c) \leq \beta_n.$$

This contradicts the assumption of Case 1 that (B_n) is critical.

Case 2: (B_k) is critical for some k such that $1 \leq k \leq m$. In this case,

$$\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_k(t_c^-) = \beta_k. \tag{18}$$

Our proof for Case 2 utilizes a number of claims, which we prove in turn.

► **Claim 10.** $\text{Lag}_k(t_c^-) \geq x_k$.

Proof. If $k = 1$, then by (18), $\text{Lag}_1(t_c^-) = \beta_1$. Also, by (X_1) , $\beta_1 = x_1$, from which $\text{Lag}_1(t_c^-) \geq x_1$ follows. The remaining possibility, $2 \leq k \leq m$, is addressed as follows.

$$\begin{aligned}
 \text{Lag}_k(t_c^-) &= \{\text{by (18)}\} \\
 &\quad \beta_k - (\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_{k-1}(t_c^-)) \\
 &\geq \{\text{since } (B_{k-1}) \text{ holds at time } t_c^-\} \\
 &\quad \beta_k - \beta_{k-1} \\
 &= \{\text{by (X}_k)\} \\
 &\quad x_k
 \end{aligned}$$



► **Claim 11.** If $k \leq m - 1$, then $\text{Lag}_{k+1}(t_c^-) \leq x_{k+1}$.

Proof. Because $k \leq m - 1$ and (by assumption) $n \geq m$, $\text{Lag}_{k+1}(t_c^-)$ is well-defined. The claim is established by the following reasoning.

$$\begin{aligned} \text{Lag}_{k+1}(t_c^-) &\leq \{\text{because } (\text{B}_{k+1}) \text{ holds at time } t_c^-\} \\ &\quad \beta_{k+1} - (\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_k(t_c^-)) \\ &= \{\text{by (18)}\} \\ &\quad \beta_{k+1} - \beta_k \\ &= \{\text{by (X}_{k+1}\text{)}\} \\ &\quad x_{k+1} \end{aligned}$$

◀

► **Claim 12.** If $k = m$ and $n > m$, then $\text{Lag}_{k+1}(t_c^-) \leq 0$.

Proof. $k = m$ implies that (18) can be re-written as

$$\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_m(t_c^-) = \beta_m. \quad (19)$$

Also, because (B_n) holds at time t_c^- ,

$$\text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_n(t_c^-) \leq \beta_n. \quad (20)$$

Therefore, given $n > m$ (from the statement of the claim), by (19) and (20), we have

$$\text{Lag}_{m+1}(t_c^-) + \text{Lag}_{m+2}(t_c^-) + \cdots + \text{Lag}_n(t_c^-) \leq \beta_n - \beta_m. \quad (21)$$

Therefore,

$$\begin{aligned} \text{Lag}_{m+1}(t_c^-) &\leq \{\text{by (21)}\} \\ &\quad \beta_n - \beta_m - (\text{Lag}_{m+2}(t_c^-) + \text{Lag}_{m+3}(t_c^-) + \cdots + \text{Lag}_n(t_c^-)) \\ &\leq \{\text{by Lemma 4}\} \\ &\quad \beta_n - \beta_m - ((-C_{\max}) \cdot (n - m - 1)) \\ &= \{\text{by (X}_n\text{) and (Y}_1\text{)}\} \\ &\quad - (n - m - 1)C_{\max} + (n - m - 1)C_{\max} \\ &= \{\text{canceling}\} \\ &\quad 0. \end{aligned} \quad (22)$$

Because $k = m$, the claim follows from (22). ◀

► **Claim 13.** If $k \leq m - 1$ or $n > m$, then $\text{Lag}_k(t_c^-) \geq \rho \cdot \text{Lag}_{k+1}(t_c^-) + C_{\max}$.

Proof. If $k \leq m - 1$, then

$$\begin{aligned} \text{Lag}_k(t_c^-) &\geq \{\text{by Claim 10}\} \\ &\quad x_k \\ &= \{\text{by (Y}_3\text{)}\} \\ &\quad \rho \cdot x_{k+1} + C_{\max} \\ &\geq \{\text{by Claim 11}\} \\ &\quad \rho \cdot \text{Lag}_{k+1}(t_c^-) + C_{\max}. \end{aligned} \quad (23)$$



If $k = m$ (recall that $k \leq m$ by the specification of Case 2) and $n > m$, then by Claim 12,

$$\text{Lag}_{m+1}(t_c^-) \leq 0. \quad (24)$$

Furthermore,

$$\begin{aligned} \text{Lag}_m(t_c^-) &\geq \{\text{by Claim 10}\} \\ &= \{\text{by } (Y_2)\} \\ &\quad (n - m + 1)C_{\max} \\ &> \{\text{because } C_{\max} > 0 \text{ and } n > m\} \\ &\quad C_{\max} \\ &\geq \{\text{by (24), and because } \rho = u_1/u_n > 0\} \\ &\quad \rho \cdot \text{Lag}_{m+1}(t_c^-) + C_{\max}. \end{aligned} \quad (25)$$

Thus, by (23) and (25) the claim follows. \blacktriangleleft

In considering the next claim, recall the following: at time t , $\text{Lag}_\ell(t)$ denotes the ℓ^{th} largest lag among all n tasks and $\mathcal{T}_\ell(t)$ denotes the set of ℓ tasks with largest lag values.

► **Claim 14.** *The k tasks in $\mathcal{T}_k(t_c^-)$ are scheduled on the k fastest processors within $[t_c^-, t_c]$.*

Proof. Because $\rho = u_1/u_n \geq 1$ and (by assumption) $n \geq m$, by (Y₂) and (Y₃), it can be shown that $x_k \geq C_{\max}$ for $1 \leq k \leq m$. Therefore, by Claim 10, we have $\text{Lag}_k(t_c^-) \geq x_k \geq C_{\max} > 0$. By Lemma 1, this implies that each of the k tasks in $\mathcal{T}_k(t_c^-)$ is pending at time t_c^- . Therefore, by Policy (G), it suffices to prove that the k tasks in $\mathcal{T}_k(t_c^-)$ have the k earliest effective deadlines at time t_c^- (with no tie with the $(k+1)^{\text{st}}$ earliest effective deadline).⁹ If $k = m$ and $n = m$, then there are k tasks in total in the system. In this case, the k pending tasks in $\mathcal{T}_k(t_c^-)$ clearly have the k earliest effective deadlines at time t_c^- .

In the rest of the proof, we consider the remaining possibility, i.e., $k \leq m-1$ or $n > m$. By Claim 13,

$$\text{Lag}_k(t_c^-) \geq \rho \cdot \text{Lag}_{k+1}(t_c^-) + C_{\max}. \quad (26)$$

By the definition of Lag , (26) implies that for any i and j such that $1 \leq i \leq k$ and $k+1 \leq j \leq n$, we have $\text{Lag}_i(t_c^-) \geq \rho \cdot \text{Lag}_j(t_c^-) + C_{\max}$. This implies that, for any task $\tau_p \in \mathcal{T}_k(t_c^-)$ and any task $\tau_q \notin \mathcal{T}_k(t_c^-)$, $\text{lag}(\tau_p, t_c^-, \mathcal{S}) \geq \rho \cdot \text{lag}(\tau_q, t_c^-, \mathcal{S}) + C_{\max}$. Therefore, if a task $\tau_q \notin \mathcal{T}_k(t_c^-)$ is pending at time t_c^- , then by Lemma 5, its effective deadline is strictly greater than the effective deadline of any task $\tau_p \in \mathcal{T}_k(t_c^-)$; if τ_q is not pending at time t_c^- , then it has no pending jobs and no effective deadline by definition. Therefore, the k tasks in $\mathcal{T}_k(t_c^-)$ have the k earliest effective deadlines at time t_c^- . The claim follows. \blacktriangleleft

⁹ Note that ε can be selected to be small enough to ensure that no scheduling event happens within the interval $[t_c^-, t_c]$, including job completions. Therefore, any task scheduled at time t_c^- will continuously execute during time interval $[t_c^-, t_c]$ on the same processor.

► **Claim 15.** $\mathcal{T}_k(t_c^-) = \mathcal{T}_k(t_c)$.

Proof. If $k = m$ and $n = m$, then there are k tasks in total in the system, so the claim clearly holds. Therefore, in the rest of the proof, we assume $k \leq m - 1$ or $n > m$, which implies that either $k \leq m - 1$ holds or $k = m$ and $n > m$ hold, by the specification of Case 2. If $k \leq m - 1$, then

$$\begin{aligned}
\text{Lag}_k(t_c^-) &\geq \{\text{by Claim 10}\} \\
&= \{x_k\} \\
&= \{\text{by (Y}_3)\} \\
&= \rho \cdot x_{k+1} + C_{\max} \\
&\geq \{\text{because } \rho = u_1/u_n \geq 1 \text{ and } x_{k+1} \geq 0\} \\
&= x_{k+1} + C_{\max} \\
&\geq \{\text{by Claim 11}\} \\
&= \text{Lag}_{k+1}(t_c^-) + C_{\max}. \tag{27}
\end{aligned}$$

If $k = m$ and $n > m$, then

$$\begin{aligned}
\text{Lag}_k(t_c^-) &\geq \{\text{by Claim 10 and because } k = m\} \\
&= \{x_m\} \\
&= \{\text{by (Y}_2)\} \\
&= (n - m + 1)C_{\max} \\
&> \{\text{because } n > m\} \\
&= C_{\max} \\
&\geq \{\text{because } \text{Lag}_{k+1}(t_c^-) \leq 0, \text{ by Claim 12}\} \\
&= \text{Lag}_{k+1}(t_c^-) + C_{\max}. \tag{28}
\end{aligned}$$

Thus, for $k \leq m - 1$ or $n > m$, by (27) and (28), we have

$$\text{Lag}_k(t_c^-) - \text{Lag}_{k+1}(t_c^-) \geq C_{\max}. \tag{29}$$

By the definition of Lag, (29) implies that for any i and j such that $1 \leq i \leq k$ and $k + 1 \leq j \leq n$, we have $\text{Lag}_i(t_c^-) - \text{Lag}_j(t_c^-) \geq C_{\max}$. This implies that, for any task $\tau_p \in \mathcal{T}_k(t_c^-)$ and any task $\tau_q \notin \mathcal{T}_k(t_c^-)$, we have

$$\text{lag}(\tau_p, t_c^-, \mathcal{S}) - \text{lag}(\tau_q, t_c^-, \mathcal{S}) \geq C_{\max}. \tag{30}$$

Therefore,

$$\begin{aligned}
&\text{lag}(\tau_p, t_c, \mathcal{S}) - \text{lag}(\tau_q, t_c, \mathcal{S}) \\
&= \{\text{by (4)}\} \\
&= \text{lag}(\tau_p, t_c^-, \mathcal{S}) + \mathsf{A}(\mathcal{I}, \tau_p, t_c^-, t_c) - \mathsf{A}(\mathcal{S}, \tau_p, t_c^-, t_c) \\
&\quad - \text{lag}(\tau_q, t_c^-, \mathcal{S}) - \mathsf{A}(\mathcal{I}, \tau_q, t_c^-, t_c) + \mathsf{A}(\mathcal{S}, \tau_q, t_c^-, t_c) \\
&\geq \{\text{since } 0 \leq \mathsf{A}(\mathcal{I}, \tau_i, t_1, t_2) \leq u_1 \cdot (t_2 - t_1) \text{ and } 0 \leq \mathsf{A}(\mathcal{S}, \tau_i, t_1, t_2) \leq s_1 \cdot (t_2 - t_1)\} \\
&\quad \text{lag}(\tau_p, t_c^-, \mathcal{S}) + 0 - \varepsilon \cdot s_1 - \text{lag}(\tau_q, t_c^-, \mathcal{S}) - \varepsilon \cdot u_1 + 0 \\
&\geq \{\text{by (30) and rearranging}\} \\
&\quad C_{\max} - \varepsilon \cdot (s_1 + u_1) \\
&> \{\text{because } C_{\max} > 0 \text{ and } \varepsilon < C_{\max}/(s_1 + u_1)\}^{10} \\
&\quad 0.
\end{aligned}$$

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Thus, at time t_c , any task in $\mathcal{T}_k(t_c^-)$ has a strictly greater lag than any task not in $\mathcal{T}_k(t_c^-)$. This implies that $\mathcal{T}_k(t_c)$ and $\mathcal{T}_k(t_c^-)$ consist of the same set of tasks. ◀

By Claim 14 and Claim 15, the k tasks in $\mathcal{T}_k(t_c)$ are continuously scheduled on the k fastest processor during time interval $[t_c^-, t_c]$, so

$$\sum_{\tau_i \in \mathcal{T}_k(t_c)} A(\mathcal{S}, \tau_i, t_c^-, t_c) = S_k \cdot \varepsilon. \quad (31)$$

Also, by (1),

$$\sum_{\tau_i \in \mathcal{T}_k(t_c)} A(\mathcal{I}, \tau_i, t_c^-, t_c) \leq \sum_{\tau_i \in \mathcal{T}_k(t_c)} u_i \cdot \varepsilon \leq U_k \cdot \varepsilon. \quad (32)$$

Therefore,

$$\begin{aligned} & \text{Lag}_1(t_c) + \text{Lag}_2(t_c) + \cdots + \text{Lag}_k(t_c) \\ &= \{\text{by Claim 15 and by (4)}\} \\ & \text{Lag}_1(t_c^-) + \text{Lag}_2(t_c^-) + \cdots + \text{Lag}_k(t_c^-) + \sum_{\tau_i \in \mathcal{T}_k(t_c)} A(\mathcal{I}, \tau_i, t_c^-, t_c) - \sum_{\tau_i \in \mathcal{T}_k(t_c)} A(\mathcal{S}, \tau_i, t_c^-, t_c) \\ &\leq \{\text{by (31) and (32), and because (B}_k\text{) holds at time } t_c^-\} \\ & \beta_k + (U_k - S_k) \cdot \varepsilon \\ &\leq \{\text{by (5) and (6); note that } U_m \leq U_n\} \\ & \beta_k. \end{aligned}$$

This contradicts the assumption of Case 2 that (B_k) is critical.

Finishing up. We have shown that both Case 1 and Case 2 lead to a contradiction. That is, none of the conditions $(B_1), \dots, (B_m)$, or (B_n) is critical at t_c . This contradicts the definition of t_c as the first time instant at which some inequality in (B) is false, i.e., such a t_c does not exist. The theorem follows. ◀

Using Theorem 8, we can easily derive a tardiness bound for every task as follows.

► **Theorem 16.** *In \mathcal{S} , the tardiness Δ_i of task τ_i is bounded as follows.*

$$\begin{aligned} & \text{If } \rho = 1, \text{ then } \Delta_i \leq \frac{n \cdot C_{\max}}{u_i}; \\ & \text{if } \rho > 1, \text{ then } \Delta_i \leq \frac{\rho^{m-1} \cdot (n-m+1)C_{\max} + \frac{\rho^{m-1}-1}{\rho-1} \cdot C_{\max}}{u_i}. \end{aligned}$$

Proof. By Theorem 8, $\text{Lag}_1(t) \leq \beta_1$ holds at every time instant t . By the definition of $\text{Lag}_1(t)$, this implies that, for each i , $\text{lag}(\tau_i, t, \mathcal{S}) \leq \beta_1$ holds for all t . Thus, by Corollary 3, task τ_i has a tardiness bound of β_1/u_i . By (X₁), $\beta_1 = x_1$, so to complete the proof, we merely need to calculate x_1 .

If $\rho = 1$ (i.e., $u_1/u_n = 1$, which implies that every task has the same utilization), then by (Y₃), $x_1 = x_m + (m-1)C_{\max}$. Thus, by (Y₂), we have $x_1 = n \cdot C_{\max}$.

¹⁰ ε can be selected small enough to ensure $\varepsilon < C_{\max}/(s_1 + u_1)$.

If $\rho > 1$, then rearranging (Y₃) results in

$$x_i + \frac{C_{\max}}{\rho - 1} = \rho \left(x_{i+1} + \frac{C_{\max}}{\rho - 1} \right).$$

By iterating this recurrence, we have

$$x_1 + \frac{C_{\max}}{\rho - 1} = \rho^{m-1} \cdot \left(x_m + \frac{C_{\max}}{\rho - 1} \right).$$

Finally, applying (Y₂) yields

$$x_1 = \rho^{m-1} \cdot (n - m + 1)C_{\max} + \frac{\rho^{m-1} - 1}{\rho - 1} \cdot C_{\max}.$$

The theorem follows. ◀

Discussion. We comment here on several aspects of the tardiness bounds in Theorem 16.

First, it may seem counterintuitive that u_i appears in the denominator of both stated bounds, because this means that the bounds are greater for tasks of lower utilization. This effect follows from our proof strategy, which involved bounding the maximum lag value in the system. We allow that any task may be the one that achieves this maximum value. Given a fixed value of lag, a lower-utilization task will tend to require greater tardiness to achieve that value of lag.

Second, the term ρ^{m-1} appears in the second bound, *i.e.*, the bound is exponential with respect to the processor count m . Despite this deficiency, we have still succeeded in establishing for the first time that tardiness under GEDF is bounded on uniform platforms. Moreover, global schedulers tend to be practical in practice (*i.e.*, have reasonable overheads) only for modest processor counts [1], so typically on large multiprocessor platforms, such schedulers are applied within clusters of processors that are not too large. In such a setting, our bounds may be reasonable.

Third, the term $\rho = u_1/u_n$ appears in the second bound. This bound could therefore be rather large if $u_1 \gg u_n$. Under a clustered-scheduling approach, this issue could be eased by using heuristics to assign tasks with relatively similar utilizations to the same cluster. Also, it might be possible to encapsulate several ordinary tasks into a higher-utilization “container” task in a way that increases u_n . However, such an approach would warrant further study.

4 Removing the Assumption (A) and Extending to Sporadic Tasks

In this section, we show that the results of the prior section can be extended to apply to sporadic task systems. We begin by showing that Assumption (A) can be removed for periodic task systems.

► **Theorem 17.** *For a given periodic task set τ , let \mathcal{S} denote a GEDF schedule that satisfies (A), and let \mathcal{S}' denote a corresponding GEDF schedule where (A) does not hold (“corresponding” means that \mathcal{S} and \mathcal{S}' include exactly the same jobs, released at exactly the same time instants). Then, no job finishes later in \mathcal{S}' .*

Proof. We prove the theorem by considering jobs inductively in deadline order. (We assume that deadline ties are broken the same way in both schedules.) Note that, by Policy (G), the scheduling of a given job is not impacted by any lower priority jobs.

Base case. The highest-priority job cannot finish later in \mathcal{S}' . In particular, because this job has the highest priority, it will execute continuously on the fastest processor once released.

Inductive step. Let \mathcal{J} denote the set of k highest-priority jobs, and assume that these jobs do not finish later in \mathcal{S}' . Also, let J denote the $(k+1)^{st}$ highest-priority job. We show that J also does not finish later in \mathcal{S}' .

Because no job in \mathcal{J} finishes later in \mathcal{S}' , at any time instant t , the number of *ready* jobs in \mathcal{S}' does not increase¹¹ in comparison to \mathcal{S} . Therefore, by Policy (G), up to any time instant t after its release, J is allocated in \mathcal{S}' no less computing capacity than in \mathcal{S} , unless J has completed in \mathcal{S}' but not in \mathcal{S} prior to time t . Finally, because J 's execution requirement is no greater in \mathcal{S}' than in \mathcal{S} , it cannot finish later in \mathcal{S}' . \blacktriangleleft

Theorem 17 implies that the tardiness bounds stated in Theorem 16 also apply to periodic tasks without Assumption (A). We now show that they apply to sporadic tasks as well.

For this purpose, we introduce a new task model, which we call the *varying-period periodic (VPP)* task model. An implicit-deadline VPP task τ_i^V has a pre-defined utilization u_i^V and also releases a sequence of jobs. However, in contrast to the ordinary periodic task model, each VPP job $J_{i,j}^V$ has its own worst-case execution requirement, denoted $C_{i,j}$. After its first invocation, a VPP task τ_i^V will release each job $J_{i,j+1}^V$ exactly $T_{i,j} = C_{i,j}/u_i$ time units after $J_{i,j}^V$'s release. Also, each job $J_{i,j}^V$ has a deadline $T_{i,j}$ time units after its release. For each VPP task τ_i^V , C_i^V is defined as $C_i^V = \max\{C_{i,j} \mid j \geq 1\}$ and T_i^V is defined as $T_i^V = \max\{T_{i,j} \mid j \geq 1\}$. Note that an ordinary periodic task τ_i is a special case of a VPP task where $C_{i,j} = C_{i,k}$ holds (and hence $T_{i,j} = T_{i,k}$ holds) for any j and k .

VPP tasks with Assumption (A). Given Assumption (A),¹² a VPP task set will have exactly the same ideal schedule as the corresponding ordinary periodic task set that includes the same tasks as in the VPP task set, where each such task has the same utilization as in the VPP task set, and releases its first jobs at the same time as in the VPP task set. Given this observation, we can re-work the proof in Sec. 3, assuming the VPP task model instead of the ordinary periodic one. In particular, every lemma, property, claim, and theorem in Sec. 3 can be re-worked to pertain to VPP tasks (with Assumption (A)) with minimal modifications, but more cumbersome notation. Specifically, C_i should replaced by C_i^V , T_i by T_i^V , and C_{\max} by C_{\max}^V , where $C_{\max}^V = \max\{C_i^V \mid 1 \leq i \leq n\}$.

The only modification that is a bit tricky is that, in the proof of Lemma 2, (8) and (9) should *not* be re-written as $0 < e_i(t) \leq C_i^V$ and $A(\mathcal{S}, \tau_i, 0, t) = W + C_i^V - e_i(t)$, respectively, because the latter might not necessarily be true even with Assumption (A). Instead, they should be re-written as $0 < e_i(t) \leq C_{i,j}$ and $A(\mathcal{S}, \tau_i, 0, t) = W + C_{i,j} - e_i(t)$, respectively. Then, (11) should be re-written as $(t - d_i(t))u_i < \text{lag}(\tau_i, t, \mathcal{S}) \leq (t - d_i(t))u_i + C_{i,j}$, which implies

$$t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} < d_i(t) \leq t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} + T_{i,j}.$$

¹¹To see this, note that, if a job J' in \mathcal{J} is not ready in \mathcal{S} but ready in \mathcal{S}' at some time t , then J' must be *pending* at time t in both \mathcal{S} and \mathcal{S}' , and some preceding job of the same task must be ready in \mathcal{S} but completed in \mathcal{S}' .

¹²That is, every VPP job $J_{i,j}^V$ executes for *exactly* its execution requirement $C_{i,j}$.

Finally, by definition, because $T_i^V \geq T_{i,j}$, we can obtain

$$t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} < d_i(t) \leq t - \frac{\text{lag}(\tau_i, t, \mathcal{S})}{u_i} + T_i^V.$$

After re-working all of the proofs in Sec. 3, the following theorem, which is the VPP counterpart to Theorem 16, can be proven, assuming Assumption (A).

► **Theorem 18.** *In \mathcal{S} , the tardiness Δ_i^V of VPP task τ_i^V is bounded as follows.*

$$\text{If } \rho = 1, \text{ then } \Delta_i^V \leq \frac{n \cdot C_{\max}^V}{u_i};$$

$$\text{if } \rho > 1, \text{ then } \Delta_i^V \leq \frac{\rho^{m-1} \cdot (n - m + 1)C_{\max}^V + \frac{\rho^{m-1}-1}{\rho-1} \cdot C_{\max}^V}{u_i}.$$

Due to the space limits, we do not prove this theorem here. As an alternative, we could have considered only the VPP task model in Sec. 3. However, this would have greatly complicated the proof in Sec. 3, both conceptually and notationally.

VPP tasks without the Assumption (A). Notice that Theorem 17 was established by reasoning only about jobs: it does not matter whether task periods are constant or variable. Thus, by Theorem 17, Theorem 18 also holds for VPP tasks *without* Assumption (A).

Sporadic tasks. We now show that tardiness bounds can be determined for any sporadic task set by viewing any instance¹³ of such a task set as an instance of a VPP task set where Assumption (A) may not hold. The transformation to a VPP task set is depicted in Fig. 1 and explained next.

In the sporadic task model, a task $\tau_i = (C_i, T_i)$ might have two consecutive jobs that have a release separation of more than T_i time units. Let $J_{i,j}$ and $J_{i,j+1}$ be such two jobs, *i.e.*, $a_{i,j+1} - a_{i,j} > T_i$. Let us denote $a_{i,j+1} - a_{i,j}$ as $(k+1)T_i + r$ where k is an integer such that $k \geq 0$ and r is a real number such that $0 \leq r < T_i$ (k and r can be easily calculated from $a_{i,j+1} - a_{i,j}$). To transform to a VPP task, we add k jobs, with the ℓ^{th} one released at time $a_{i,j} + \ell \cdot T_i$ for $1 \leq \ell \leq k$, all with a worst-case execution requirement of C_i , plus an additional job, if $r > 0$, released at time $a_{i,j+1} - r$, where this job has a worst-case execution requirement of $r \cdot C_i / T_i$. We do this for every task τ_i whenever $a_{i,j+1} - a_{i,j} > T_i$ holds for two consecutive job releases $a_{i,j}$ and $a_{i,j+1}$. The resulting task set a VPP task set, where each VPP task τ_i^V has the parameters $u_i^V = C_i / T_i$, $C_i^V = C_i$, and $T_i^V = T_i$. Now, to complete the transformation, we can simply define the *actual* execution requirement of each added job to be zero.

As we discussed earlier, Theorem 18 holds for VPP tasks without Assumption (A). Thus, the tardiness bounds in Theorem 18 apply to every instance of a VPP task set resulting from the transformation process above, and therefore to every instance of the original sporadic task set. By substituting the notation pertaining to the original sporadic task set into Theorem 18, it follows that the tardiness bounds in Theorem 16 hold for sporadic tasks as well.

¹³ An instance of a task set is defined by a set of *concrete* job release times and *actual* execution requirements that satisfy the specification of each task.

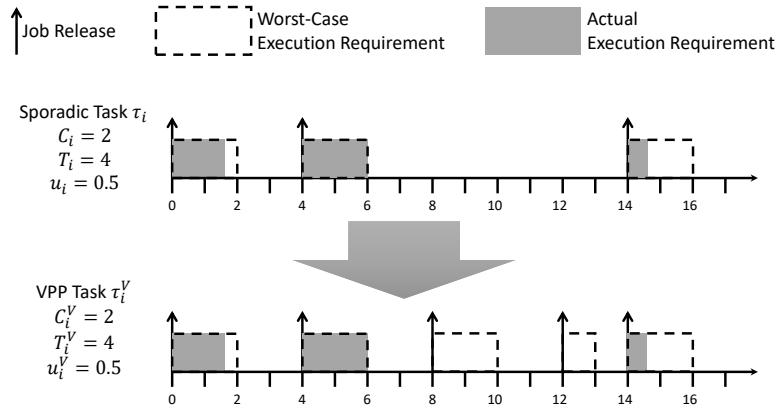


Figure 1 Transforming a sporadic task into a VPP task.

5 Conclusion

In this paper, we have shown that the GEDF scheduler is SRT optimal on uniform multiprocessor platforms, closing a longstanding open problem. Together with prior results establishing the *non*-optimality of NP-GEDF in this context [13], we now have a complete picture concerning the SRT optimality of both preemptive and non-preemptive GEDF on uniform platforms.

We established the SRT optimality of GEDF in the considered context by deriving tardiness bounds for any sporadic task system that is feasible according to the constraints specified in (5) and (6). The key to this optimality result proved to be finding a way to work these constraints into the argument. The need to exploit the constraints in (6) in particular resulted in the necessity for a proof strategy that is very different from that which had been employed previously in similar work pertaining to identical platforms [3, 9].

The establishment of tardiness bounds under GEDF on identical platforms [3] led to follow-up work in which such bounds were derived for a broader class of schedulers (particularly, “window-constrained” schedulers) [9] and for schedulers that must function in settings where processor supply is only partially available [8, 9]. In future work, we plan to consider similar extensions to the analysis presented in this paper. We also plan to investigate whether tighter tardiness bounds can be obtained generally and whether such bounds can be further honed if processors of only two different speeds exist.

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