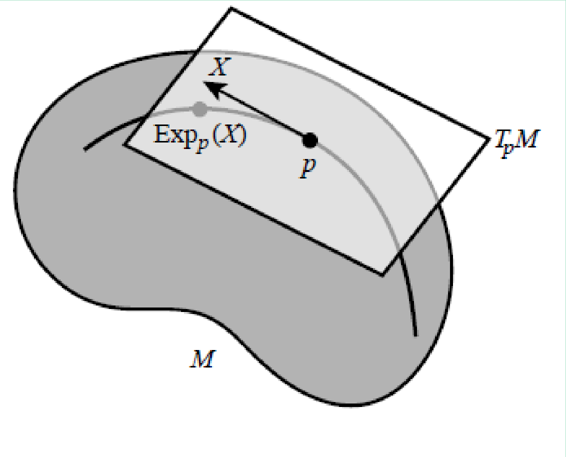
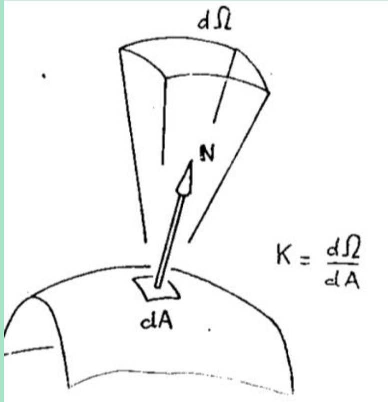
Summary:
Shape Representations and Statistics

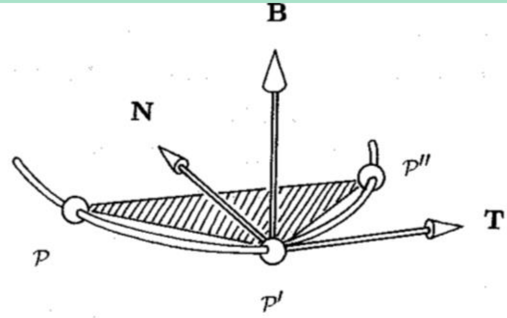


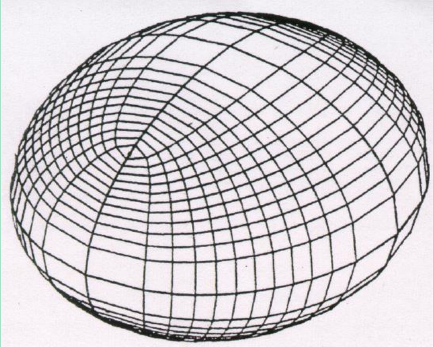
- Shape Representations
  - Local Properties
  - Shape Spaces
  - Object vs. Diffeomorphism representations
    - Multi-entity objects, shape over time
- Shape statistics
  - Over Euclidean spaces
    - PCA, DWD, Permutation methods of hypothesis testing
    - Over diffeomorphism momenta
  - Euclideanization, esp. PNS, log (positive feature)
  - Over curved manifolds
  - Correspondence

Local Properties

- **Normal directions and tangent directions**
- Fitted frames to boundary
  - Later: to interior via s-reps onion skins
  - Space curves: Frenet frames, curvature, torsion
- Curvatures: curves and surfaces
  - Esp. vertices and crests
  - Curvedness (C) and shape type (S)
- **Manifolds and geodesics**
- Distance measures
  - Riemannian metrics
  - Metric tensor: M_{II}

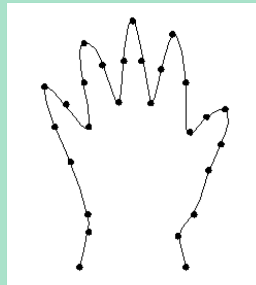


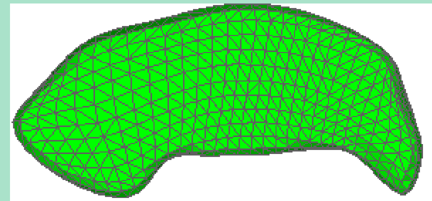


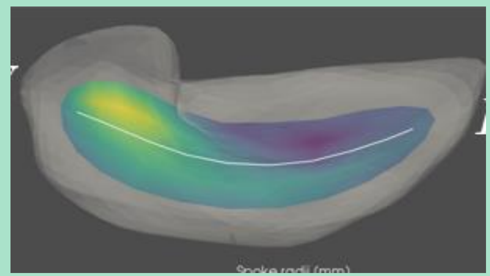


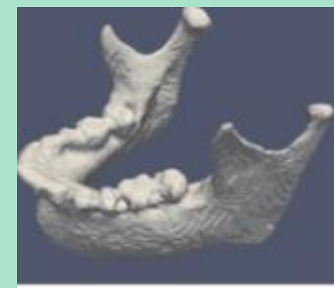
Shape Representation Categories

- Landmarks
- **Objects**
  - Boundaries
    - **Points**
    - **Normals**
    - Spherical harmonics
    - Signed distance images

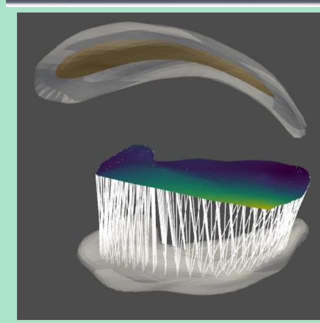




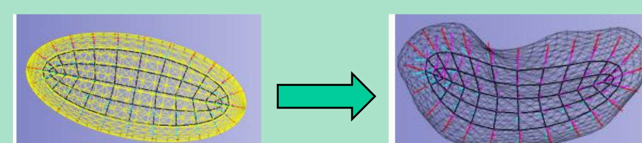




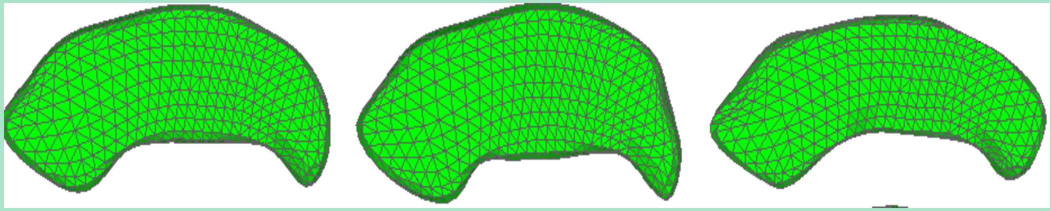
  - **Skeletal models**
    - Multifigure models
  - Landcurves: currents
  - Multi-object representations



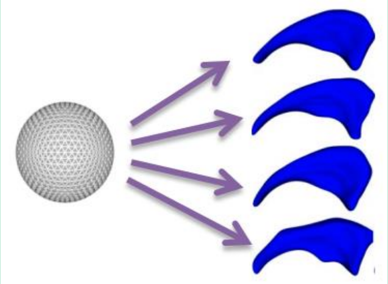
- **Diffeos from a central example**
  - From boundaries of mean
  - From s-reps of ellipsoid

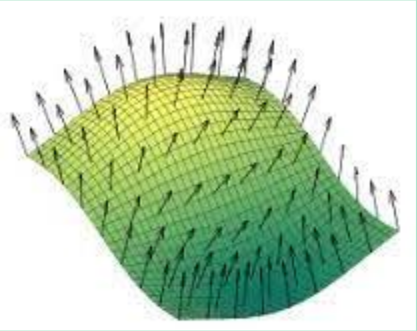


Shape Representation by Boundary Points



- Points in correspondence (PDM); or Meshes
  - Correspondence produced by
    - Diffeomorphisms
    - Skeletal models
    - Entropy minimization

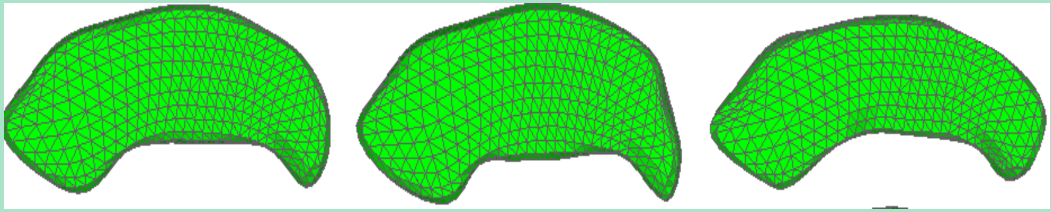




- Spherical harmonics
- Points with on landcurves (Currents)
- Normals with correspondence mod-ed out
- Signed distance images, esp. for 3D visualization
- Alignment by Procrustes
- Aligned PDMs on high-dimensional sphere

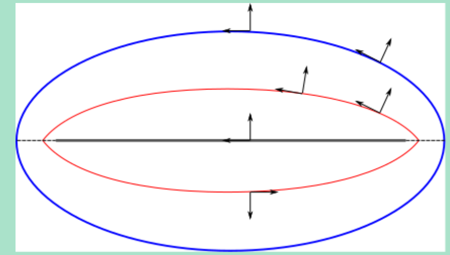
Shape Representation Designed for Correspondence

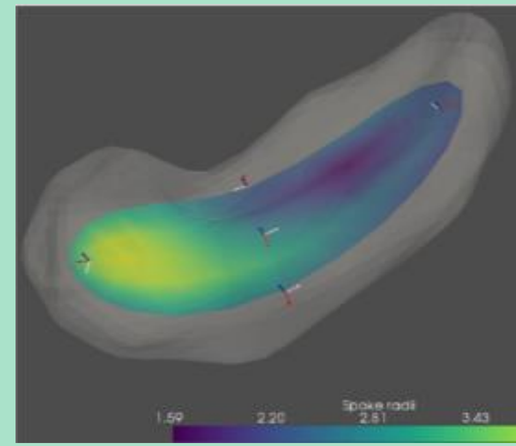
- Normals with correspondence mod-ed out [Srivastava, Kurtek]
- Skeletal models fit from ellipsoid
  - Interior positions correspondence
- ?? Diffeomorphisms based on boundaries
  - Points
  - Curves, e.g., crests

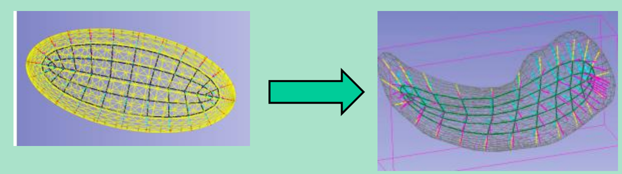


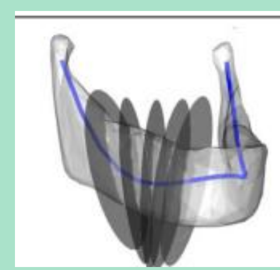
Shape Representation by Skeletal Models

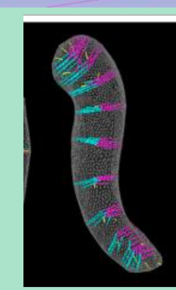
- Medial and skeletal mathematics
  - Blum medial axis: bitangent spheres
    - Geometric relations among axis and width
    - Singularities: branching, ends, etc.
    - Radial shape operator S_{rad}
    - Radial distance
      - Geometry of onion skins
    - **Cm-reps** based on Blum math
- Skeletal generalization: **S-reps**
  - Skeleton and spokes
  - Discrete s-reps
  - Deformation from ellipsoids
    - Alignment-free coordinates
    - Fitted frames
  - Slabular planar cross-section sweeping
    - Taheri s-reps: spine





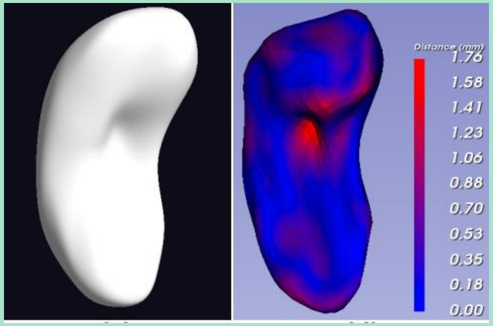






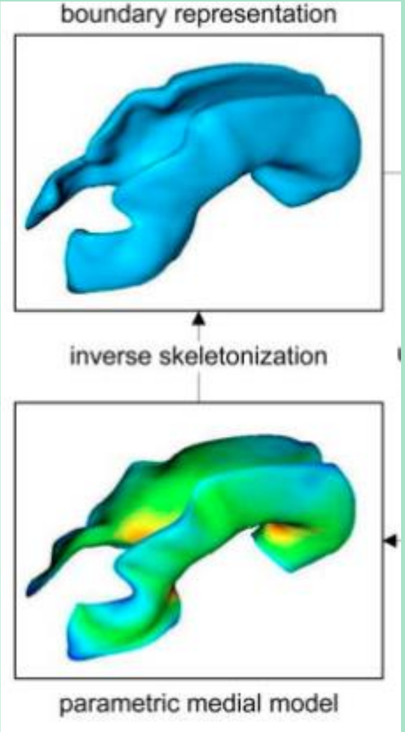
Shape Representation by Skeletal Models

- S-reps
  - Fitting to boundaries
    - Optimization



Bdry implied by s-rep — Target object

- Cm-reps [2 lectures by P. Yushkevich]
  - Based on mesh and Blum conditions
    - **Explicit: inverse skeletonization using biharmonic PDE**
    - **Implicit: deformation of boundary & medial locus preserving medial linkages to boundary**
      - Like s-reps, starts from model with known branching topology and medial locus
    - Based on splines in <u>x</u> and width



Euclidean Statistics Methods

- Means
- PCA: feature reduction and removal of noise
- Classification
  - Producing separation direction
  - Multi-entity analysis: DIVAS
- Hypothesis Testing
  - Permutation tests
  - Corrections for multiple tests
- Segmentation by posterior optimization
  - Priors via shape representation statistics
  - Likelihoods via shape-based coordinates
- Longitudinal methods

Euclidean Statistics Methods, 2

- PCA: feature reduction and removal of noise
  - Eigenanalysis of covariance
  - Features via inner product with eigenvectors
  - And other related producers of modes of variation
- Classification
  - Producing separation direction
    - Distance-Weighted Discrimination
      - Σ 1/r + Σ misclassification penalties
    - Kernels, esp. radial basis functions
    - Multi-entity analysis: DIVAS
  - Producing & using histograms along separation direction

Euclidean Statistics Methods, 3

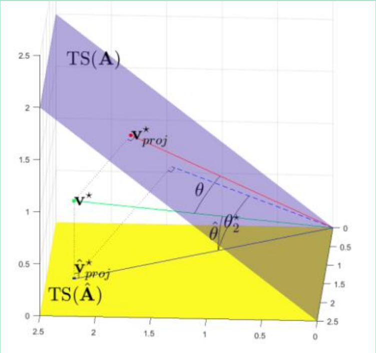
- Hypothesis Testing
  - Permutation tests
  - Corrections for multiple tests
- Segmentation by posterior optimization
  - Priors via shape representation statistics
  - Likelihoods via shape-based coordinates
  - S-reps provide both
- Longitudinal methods
  - Variations on the Euclidean space
    - Curves over t for intra-subject distances
    - Inter-subject distances between intra-subject curves
    - Generalized linear models
  - Diffeomorphisms’ momenta
    - Intra subject diffeos over t
    - Inter-subject diffeos

Applicability of Marchenko-Pastur Analysis on PNS modes [Choi]

- Our "eigenanalysis" is via PNS, i.e., on sphere
- Hyo Young Choi studied this problem
  - By producing derived features from the sphere that do follow the MP distribution
    - But she only studied great subspheres, whereas the common analysis uses small subspheres, when hypothesis test supports But the open problem remains: How to select the eigenmodesde by depolluting the MP-plot, given noise properties from high eigenvalues which show ~pure noise
  - Besides dealing with the actual subsphere approach

Euclidean Statistics Methods, 4

- Multi-entity statistics (DIVAS, AJIVE)
  - Noise removal via PCA of each entity
  - Space of subjects: tuple of noise-removed features based on PCAs
    - Subspaces each specified by orthogonal linear combinations of subjects-space features
  - Subspace of joint features via Principal Angle Analysis

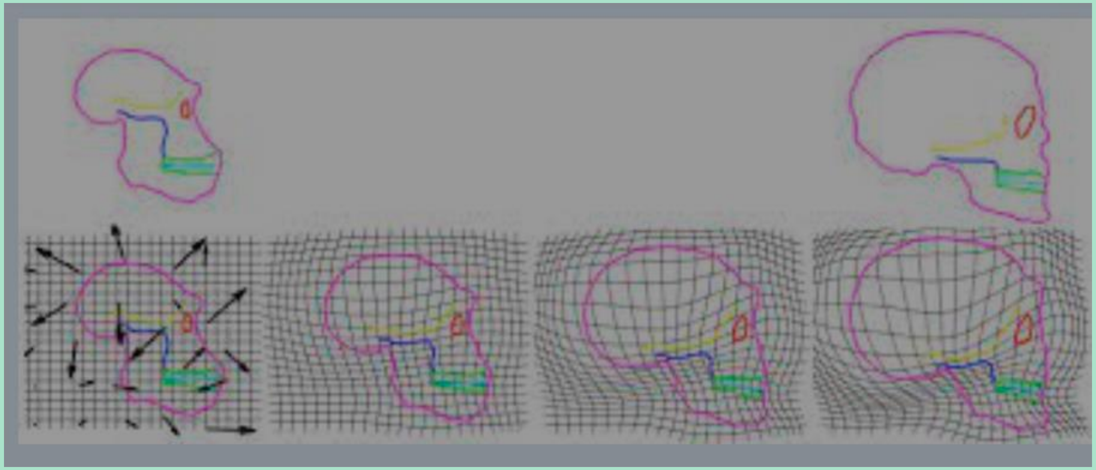


from Prothero et al.

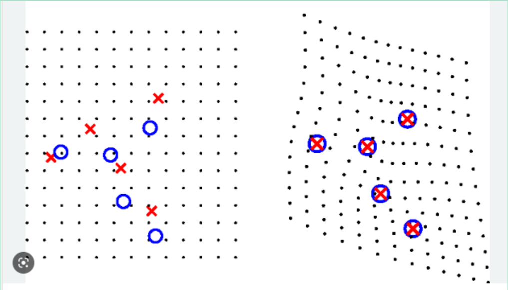
  - Subspaces of individual features for each individual
    - Orthogonal to joint subspace
    - Not necessarily orthogonal to subspace of other individuals

Shape Representation by Deformations

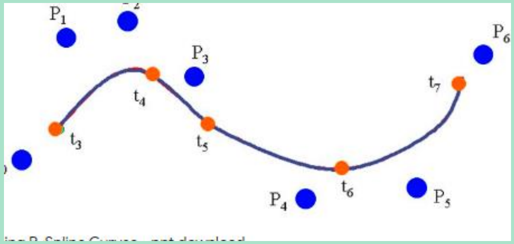
- Diffeomorphisms: velocities
  - Points data
  - Currents data
    - For landcurves
    - For surfaces



- Displacements
  - Thin-plate splines



  - B-splines
  - Elastic deformations



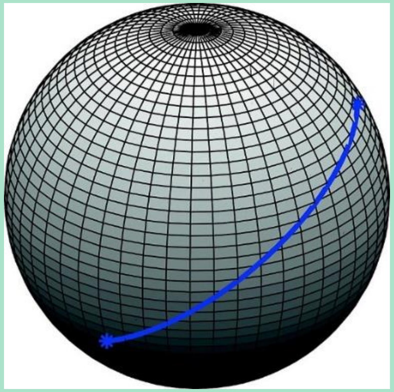
Thin Plate Splines Method

- Fast: based on a solution to linear equations
  - Typically preceded by optimum affine transformation
- Elastic warp in each variable
  - $\mathbf{\underline{x}'(\underline{x})=\underline{c}+A\underline{x}+\Sigma_j \underline{w}_j U(|\underline{x}-\underline{x}^j|)}$
  - Basis functions $U(|\underline{x}-\underline{x}^j|)$ depend on moving image's landmarks $\underline{\mathbf{x}}^{\mathbf{j}}$
    - Radial bases: $U(d) = d^2 \log d$ for 2D, d^3 for 3D
- Solve linearly for c, A, $\{\underline{\mathbf{w}}_{\mathbf{j}}\}$ based on $\{\underline{\Delta \mathbf{x}}^{\mathbf{j}}\}$
- Minimizing Frobenius norm: $\int^{\infty \text{ space}} \Sigma_{all}$ 2nd partial derivatives^2, so smooth
  - 27 terms for 3D: 9 for $\Delta x(x,y,z)$, 9 for $\Delta y(x,y,z)$, 9 for $\Delta z(x,y,z)$
- Not necessarily diffeomorphic; may produce folding
  - Normally OK if displacements << inter-landmark spacing
- Not symmetric, not affine invariant

Large Deformation Diffeometric Metric Mapping (LDDMM) Methods

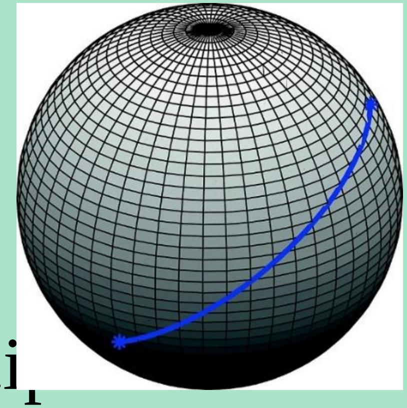
- Consider the shape space of diffeomorphisms
- Let metric on that space measure spatial smoothness within a velocity image
- We want the shortest geodesic from Identity mapping to the diffeomorphism that maps the corresponding points onto each other
- Typically requires iterative optimization
- Implementations
  - Deformetrica
    - Can also use corresponding space curves
  - Joshi

Statistics on Curved Manifolds
[Esp. Fletcher lecture]



- Exp_p and Log_p
- Fréchet and backwards means
- Geodesics
  - And other polynomial generalizations
- Shape spaces, esp. spheres and polyspheres: PNS
- Classification via separating directions
  - Esp. DWD
    - Advantage over SVM
- Longitudinal statistics
  - See later: longitudinal stats via diffeomorphisms

Statistics on Curved Manifolds, 2
[Esp. Fletcher lecture]



- Geodesics
  - Representation by point and direction
  - Generalization of line in Euclidean space
  - Yielding distances, thus Fréchet mean, princi directions
  - Polynomial generalizations
- Shape spaces, esp. spheres and polyspheres: PNS
- Classification via separating directions
  - Esp. DWD
    - Advantage over SVM
- Longitudinal statistics
  - Like Euclidean, but using geodesics, etc.
  - See later: longitudinal stats via diffeomorphism

Statistics in Shape Spaces

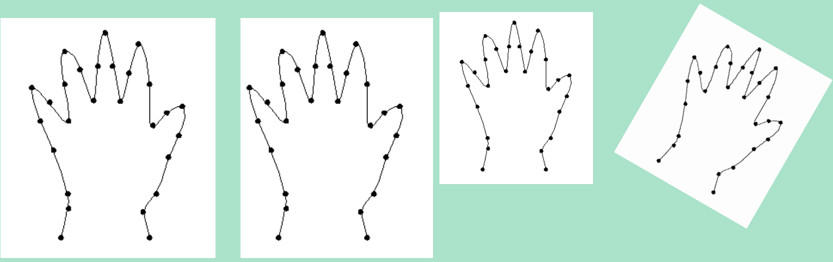
- Commensuration: scaling and weighting
- Euclideanization
  - Positive scalars
  - Directions
  - Normalized PDMs
- Directly on curved manifold

Gaussians

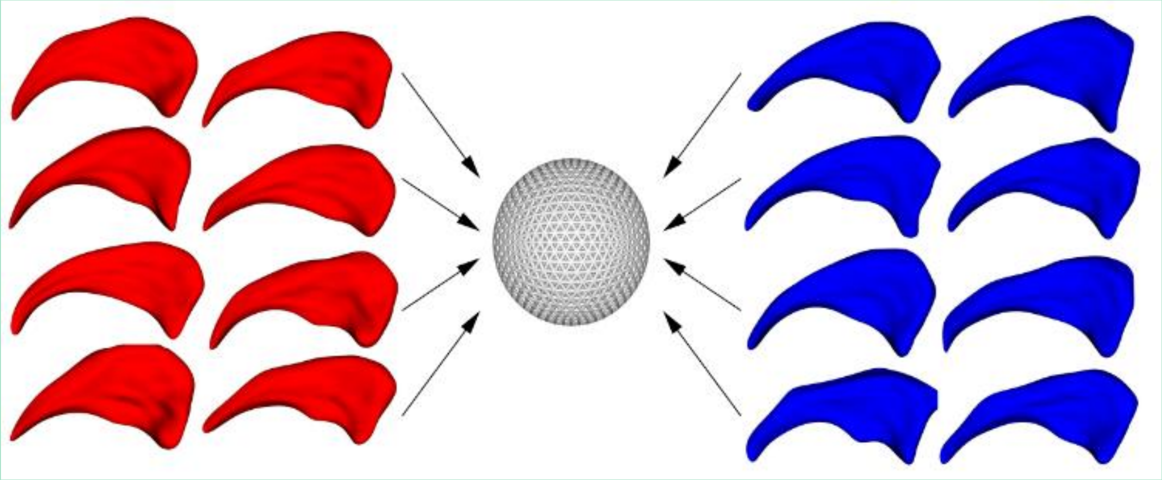
- Importance
- Relation to PCA
- Production
  - Analytic form in Euclidean space
    - Given Principal frame R and eigenvalues Λ
  - Diffusion: $\partial f(\underline{x},t)/\partial t = \nabla^2 f$ with $f(\underline{x},0)=\delta(\underline{x})$ with $t=\sigma^2/2$
  - Brownian motion
- **On curved surfaces, esp. spheres**
  - Wrapped Gaussian
  - von Mises distribution
    - ~ wrapped Gaussian on sphere
    - With an analytic form not needing sums over wraps
    - Most commonly used due to its rather simple form
  - Brownian motion (random walks)
    - Moving frames [Sommer]

Statistics on PDMs

- Procrustes alignment
  - Centering
  - Scaling via Σ squares
  - Rotation via 2nd moments

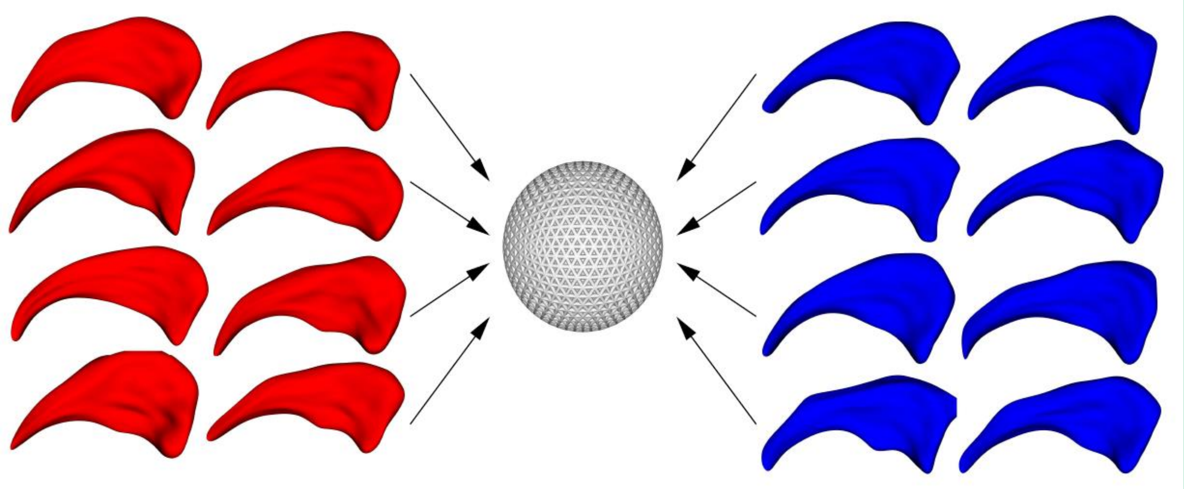


- Principal nested spheres for feature reduction
- Transformation to spherical harmonics coefficients



Statistics on PDMs Transformed into Spherical Harmonics Coefficents

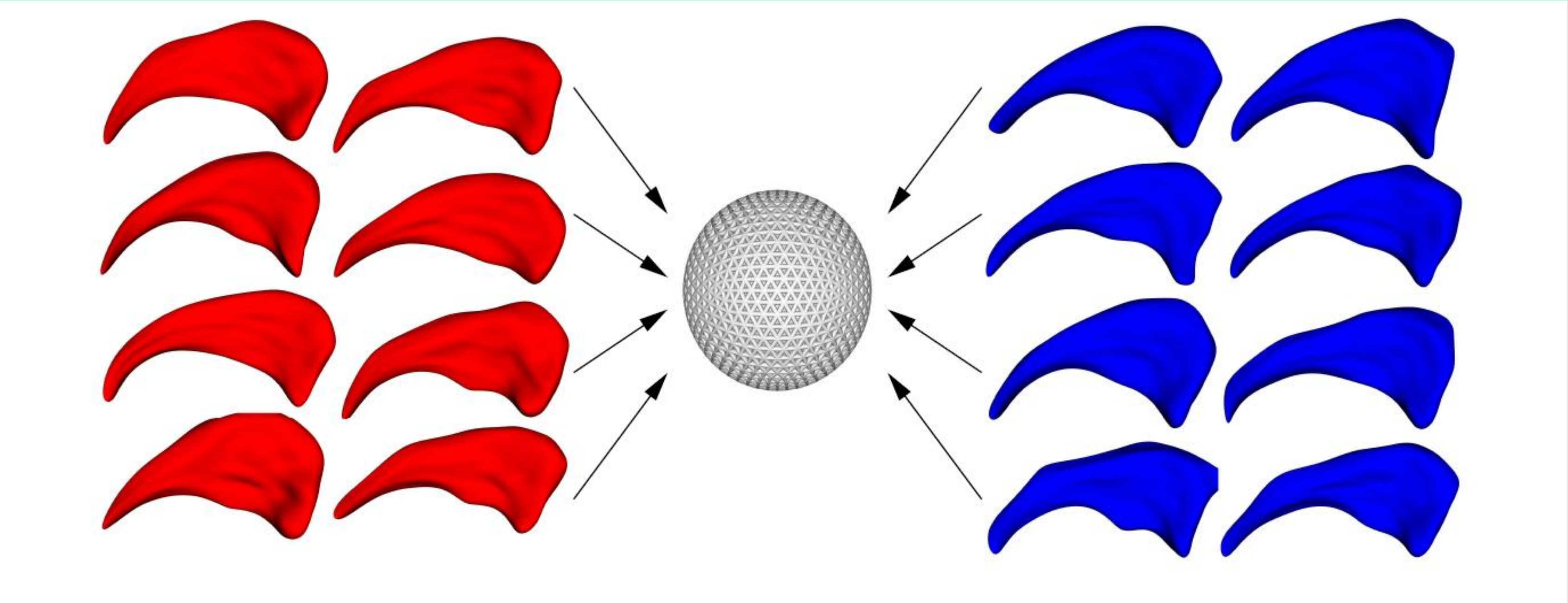
- Each object mapped from sphere: $\underline{x}(\theta,\phi)=\Sigma_i \underline{b}_i \ \psi^i(\theta,\phi)$
  - Discretized with equal area spherical triangles
  - Can do Euclidean statistics of the $\underline{b}$ values over a population
  - $\underline{b}$ values are determined globally



- Object features: coefficients of basis functions on the sphere
  - Basis functions organized by frequency in latitude and longitude
  - From $\underline{x}(\theta,\phi)$, coefficients easily obtained by dot product w/ basis
  - For <u>any</u> (θ,ϕ), $\underline{x}(\theta,\phi)$ (e.g., mean) can be computed from coefficients
    - **Correspondence via (θ,ϕ)**, but empirically not always adequate

How to get the point coordinates on the object onto the sphere

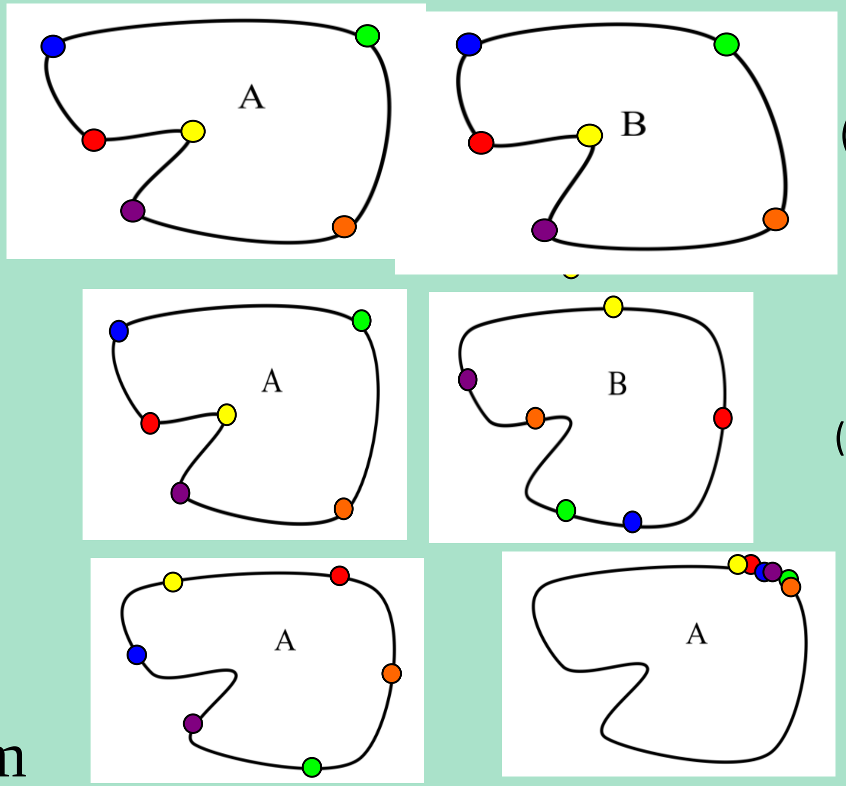
- Equal area mapping [Brechbuehler]
  - Or alternative of conformal mapping, which is angle preserving
- North pole and Greenwich meridian via best fitting ellipsoid
  - Might need straightening as a preprocessing



- Possible use of s-reps implied spacing

Correspondence

- Approaches
  - Via entropy: produce tightest ensemble p(***x***)
    - Possibly also including C, S as features [Oguz]
  - Registration
    - Via landmarks
      - thin plate splines
      - diffeo guaranteeing methods
    - Via transformation from basic object reflecting richer geometry, such as skeletal



Tight distr’n (low ensemble entropy)

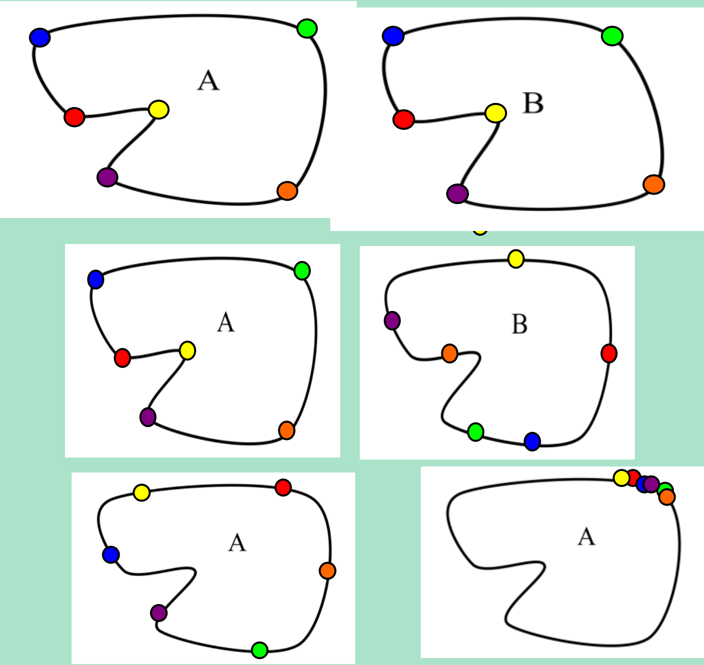
Non-tight distr’n (high ensemble entropy)

High surface entropy

Low surface entropy (uniformity)

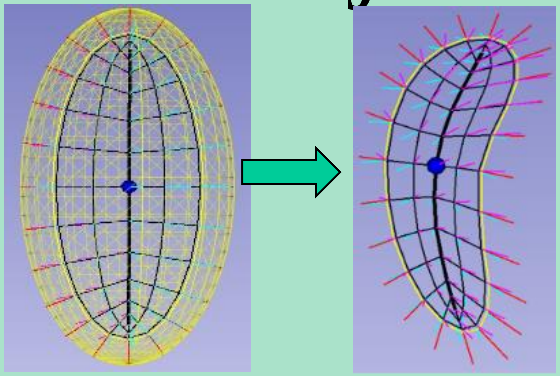
Correspondence via Entropy of PDMs

- Shapeworks [Cates, Whitaker]
  - Ensemble entropy H(ensemble) should be low (p(**x**) tight)
  - Entropy H(point positions along boundary for each case) should be high (uniformly distributed)
  - So $\min_{\underline{\mathbf{x}}}$ [$H_{\text{training cases}}$(geometry) – $\Sigma_{\text{training cases}}$ H(points on training case)]
  - Entropy via PCA: H(nD Gaussian) = (n/2) [1+ ln(2π) + avg ln λ]
  - Optimize by successively doubling number of points
    - Slow and often finds local optimum

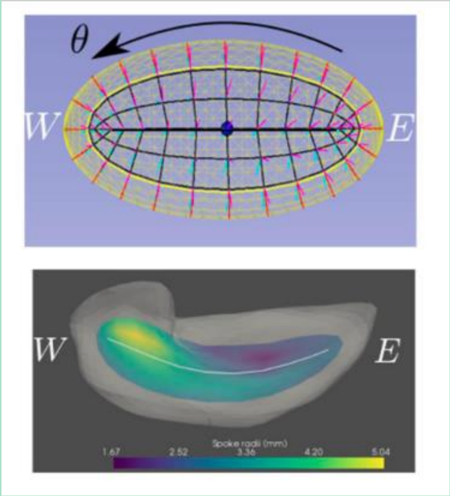


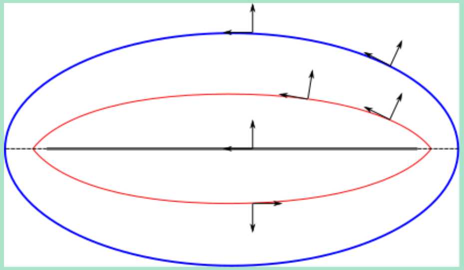
Correspondence via Skeletal Mapping from Ellipsoid to Object

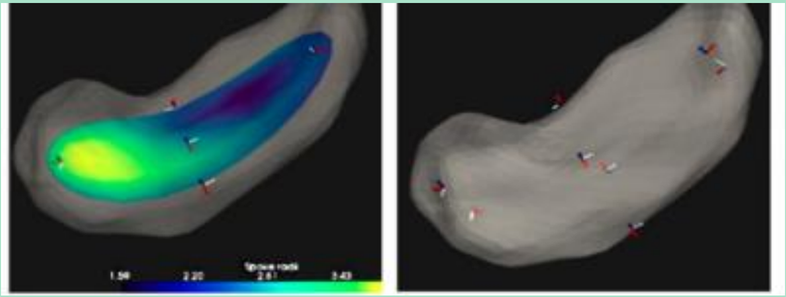
- Mapping via diffeomorphism such that
  - Vertices, crests map onto vertices
  - Crests map onto crests, crests
  - Straight spokes, radial distances map onto straight spokes, radial distances



- Defines a fitted frame at every sampled spoke point
  - i.e., on onion skins







- [Pizer, *Skeletons, Object Shape Statistics*, Frontiers in Computer Science, 2023, on google drive for Pizer, Comp 790-6

Srivastava boundary geometry modulo correspondence

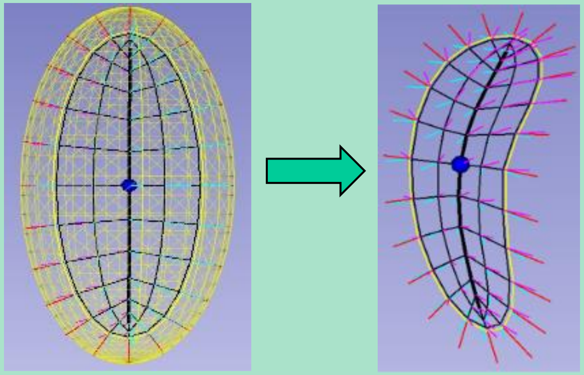
- Want representation independent of boundary parameterization
  - So inter-object distances are between equivalence classes over alignment in R^n and reparameterization should be
  - $\|[q_1],[q_2]\| = min_{O \in SO^n, \gamma \in \Gamma}\ d_c(q_1, O(q_2 \circ \gamma))/\sqrt{\dot{\gamma}}$,
    - ||A,B|| = distance^2 between A and B
    - $[q_i]$ = equivalence class of Γ = rep'ns q_i that are reparameterizations γ of boundary of object i
    - d_C is L^2 norm on boundary representation (normal)
    - *O* is orbit over reparameterizations
  - = $min_{O \in SO^n, \gamma \in \Gamma}\ d_c(q_1, O(q_2 \circ \gamma))/$ √κ(γ)

Kurtek 2D boundary geometry in 3D modulo correspondence, 3

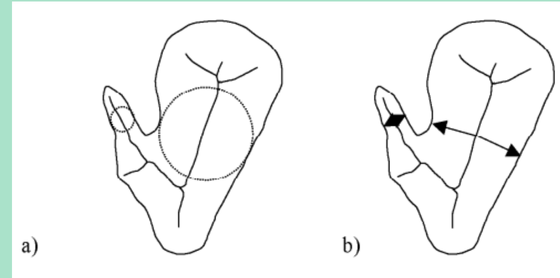
- Want representation independent of boundary parameterization
  - So inter-object distances are between equivalence classes over alignment in R^n and reparameterization
  - $d([q_1],[q_2]) = \min_{O \in SO(3), \gamma \in \Gamma} || q_1 - O_\gamma q_2 ||$
    - d(A,B) = distance between A and B
    - $[q_i]$ = equivalence class of Γ = rep'ns q_i that are reparameterizations γ of boundary of object i
    - O_γ is orbit over reparameterizations
    - In q_1 and q_2 scaled versions of the objects are used
    - Minimization of the reparameterizations needs to use its Jacobian, which captures the geometry through its fitted frames

Skeletal Representations

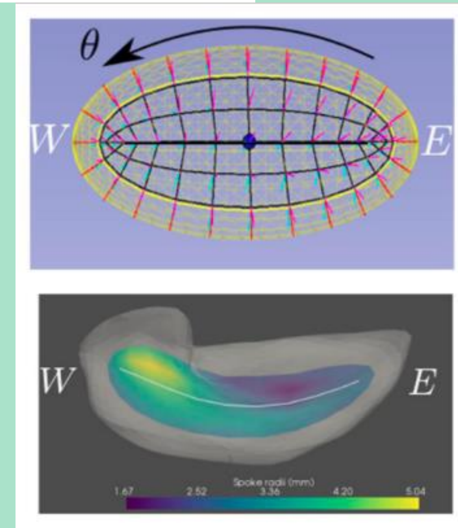
- Conceptually, skeletal representations have the following major advantages over other representations:
  - For correspondence reflect
    - Object width
    - Curvature and direction of the object *interior*
  - Ideally, capture division of object into a tree of protrusions and indentations
  - Separate width and bending features

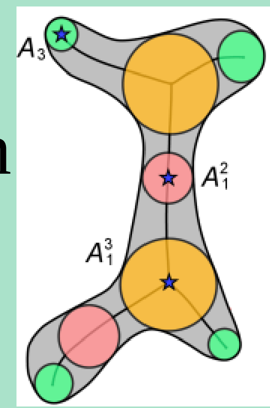


Medial and Skeletal Mathematics

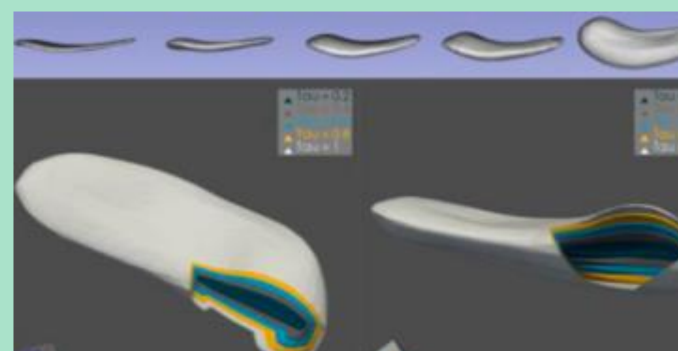


- Blum Medial representation
  - Skeleton (medial locus) $\underline{x}(u,v)$, spoke length $r(u,v)$
    - Bitangent spheres entirely in object interior with centers $\underline{x}$, radii r
    - Implies spoke $r\mathbf{U}$, orthogonal to boundary where $\mathbf{U}$ gives spoke direction
      - Computed by grassfire



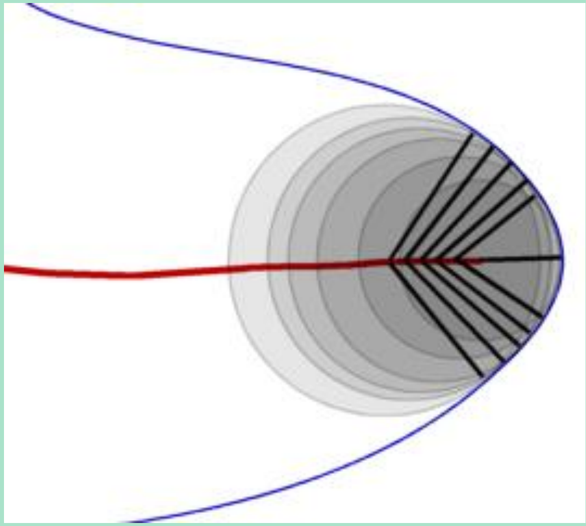


- Folded skeleton: (u,v) spherical
- Interior and boundary positions are parameterized by figural coordinates (u,v,τ_2), with τ_2 = (radial distance) fraction of spoke from $\underline{x}(u,v)$
  - $\underline{b}_{\tau 2} = \psi(\underline{x},\tau_2) = \underline{x} + \tau_2 r\mathbf{U}$
- Branch (A_1^3) and endpoints (A_3)



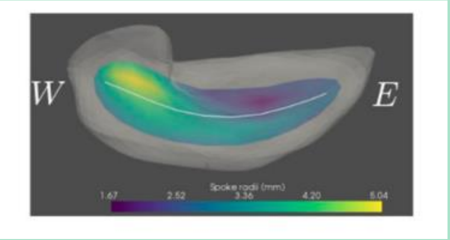
Blum Ends (Folds) in 2D

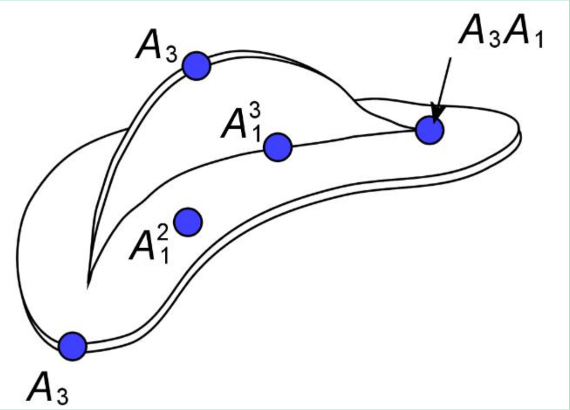
- Fold (end) atom
  - Zero object angle
  - Multiplicity 3 tangency, i.e., osculation
  - $\theta=0$, so $dr/dx = -\cos(0) = -1$
  - **Infinitely fast spoke swing in limit**

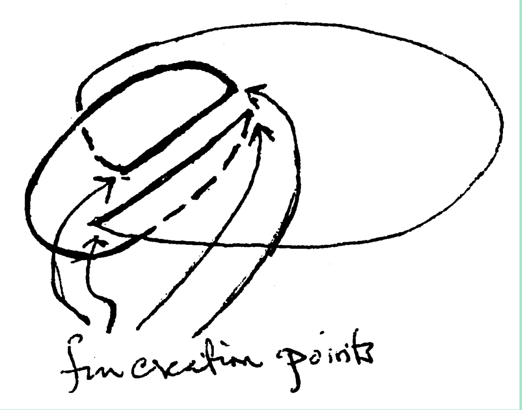


Medial Mathematics in 3D

- Medial representation singularities
  - Normal point (no singularity; 2D or 3D): A_1^2 (bitangent)
  - Point on branch curve (point for 2D): A_1^3 (tritangent)
  - Point on fold curve (point for 2D): A_3 (tangent of order 3 at 1 point)
    - Surprisingly 3rd order touching at crest
  - 4 point contact not generic in 2D but is generic in 3D: A_1^4
  - Ends of branch curves in 3D mix normal point and fold of branch : A_1A_3

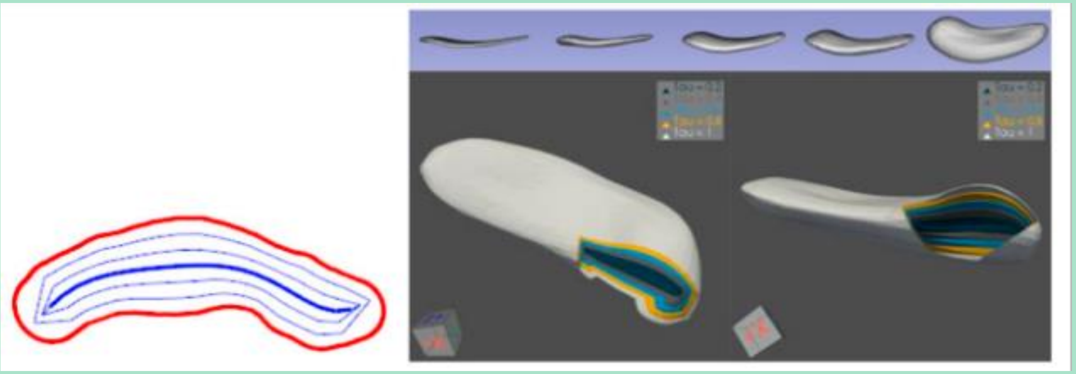






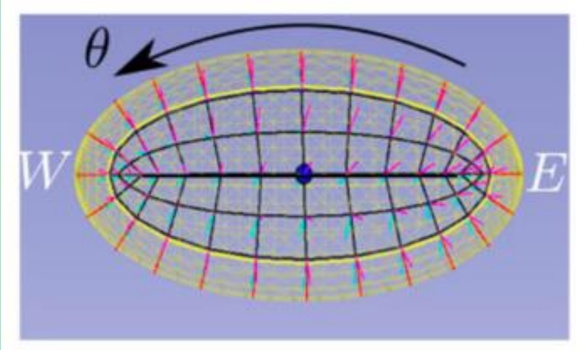
Radial Shape Operator [Damon]

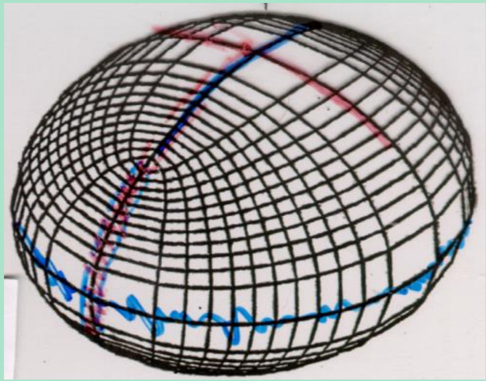
- S_{rad} = 2×2 matrix of negative of orthogonal ($\mathbf{e}_u$, $\mathbf{e}_v$) coefficients for walking directions
  - S_{rad} <u>w</u> analyzes swing of spoke direction **U** for any walking direction <u>w</u> on the skeletal tangent plane
- Radial curvatures κ_r are eigenvalues of S_{rad}
  - $r < 1/\kappa_{r_i}$ for all positive radial curvatures and all skeletal points to prevent spoke crossing in closed object interior
  - Boundary curvatures: $\kappa = \kappa_r /(1- \kappa_r)$
  - Similar formula for onion skins (which have same skeleton)



Among Objects with with Spherical Topology, The Ellipsoid: The Primordial Shape

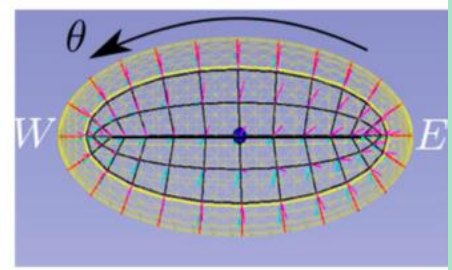
- Ellipsoid with principal radii $r_x > r_y > r_z$ is simplest shape with a skeleton in the form of a folded surface
  - Blum skeleton is ellipse in (x,y) plane with principal radii: $(r_x^2 + r_z^2)/r_x$ in x direction ; $(r_y^2 + r_z^2)/ r_y$ in y direction
  - **Crest of boundary**
    - **Medial spokes are from fold**
    - Relative max of κ_1 in $\mathbf{p}_1$ direction
    - On ellipsoid is an ellipse in the (x,y) plane
      - It has two opposing vertices

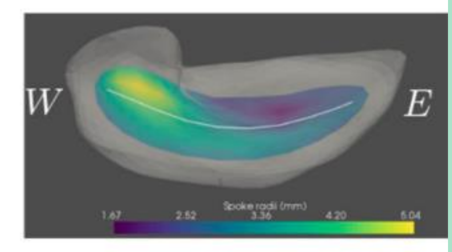


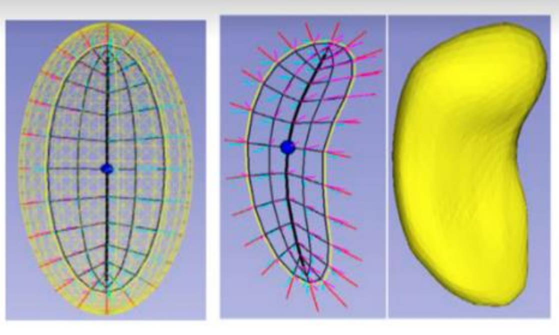


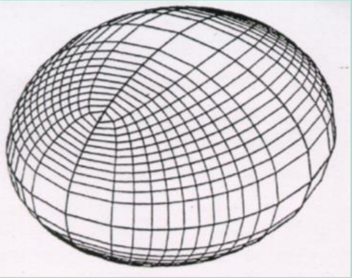
Objects with Spherical Topology and No Protrusions or Indentations

- Can be understood as diffeomorphism of the ellipsoid
  - It will have at least two opposing vertices and at least one closed crest
- Want to carry all the basic skeletal geometry into the object throughout the diffeomorphism
- **Designed to yield a strong correspondence in and near objects' interiors across objects in a population via (θ, τ_1 , τ_2)**



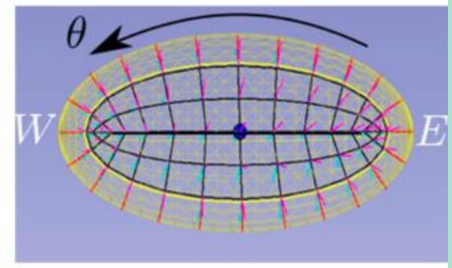


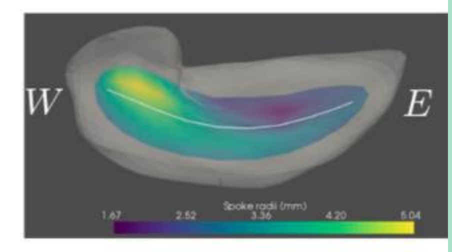


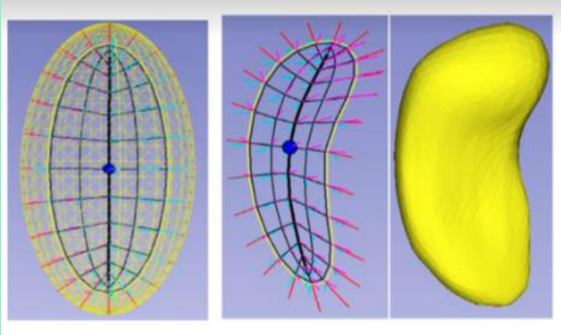


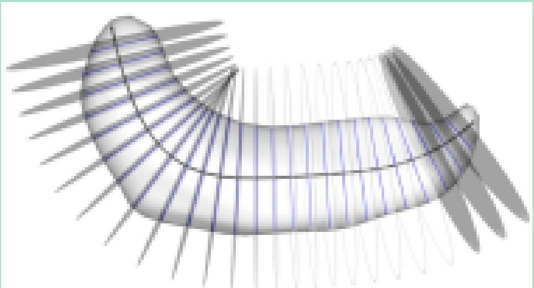
Single Figure Objects with Spherical Topology via Diffeomorphism of Ellipsoid

- For correspondence, want the diffeomorphism to carry the basic skeletal geometry of the ellipsoid into the target object
- Because the skeleton is designed to carry the curvature of the interior of the object, it appears not possible for the spokes across the skeleton (with their radial distance τ_1), which are straight in the ellipsoid's skeleton to remain straight in target object skeleton.
  - But in Taheri's swept plane skeleton coplanar spokes from skeleton with common spine point





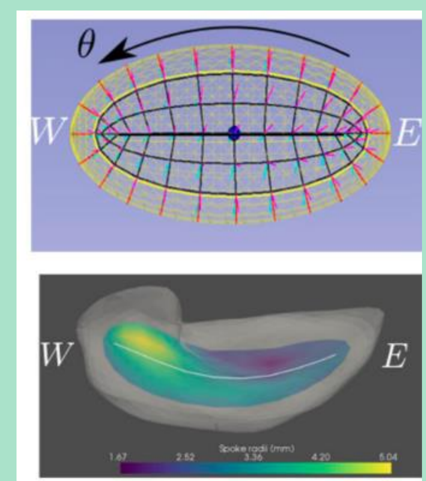




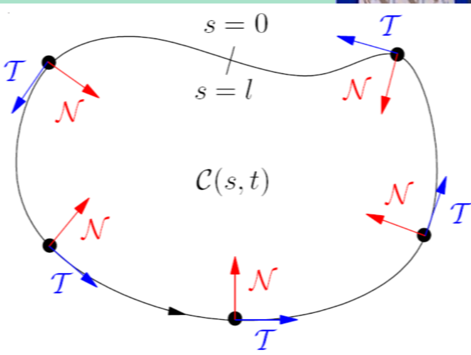
Single Figure Objects with Spherical Topology via Diffeomorphism of Ellipsoid, 2

- Computation of diffeomorphism from ellipsoid to object can be initialized with curvature-smoothing flow of target object boundary, which will approach ellipsoid

  - Before it approaches its limiting sphere
  - By **conformalized mean curvature flow** [Kazhdan]
    - But produces poor correspondence for skeletal geometry maintenance
  - Will collapse protrusions and indentations early (see subfigure discussion)
  - Its inverse, the desired diffeomorphism, needs to be modified to maintain the basic ellipsoidal skeletal geometry for the object

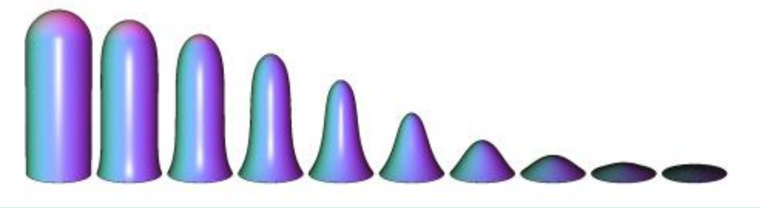


Conformalized Mean Curvature Flow of an Object Boundary



- Original idea was mean curvature flow:
  - d<u>b</u>/dt = H(<u>b</u>, t) **N**(<u>b</u>, t)
    - t is time of flow
  - Though it does deform boundary into a near ellipsoid, it collapses regions of high curvature into a point,
    - i.e., has singularities

- Improved method does not have singularities: conformalized mean curvature flow [Kazhdan]



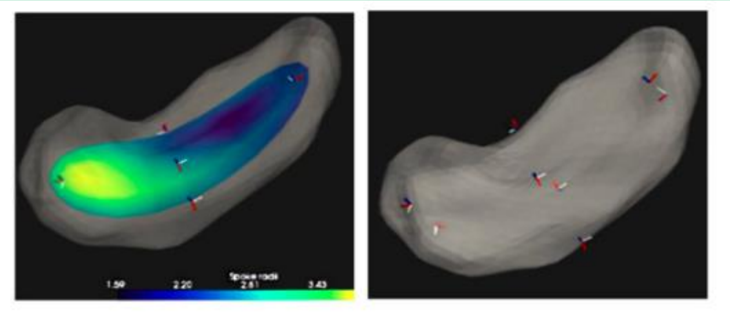
  - Changes the metric for the flow:

$$\tilde{g}_t = \sqrt{\left|g_0^{-1} g_t\right|}\, g_0.$$

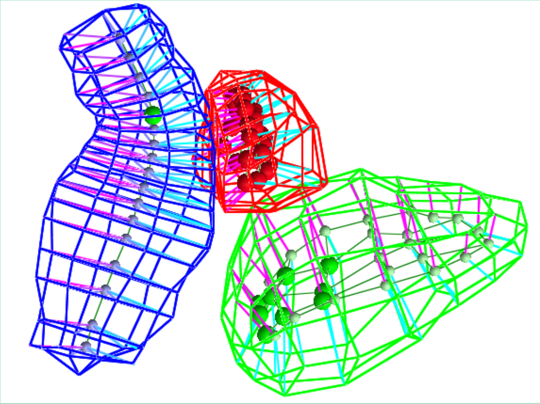
    - Metric is in principal coordinates
    - Metric changes with deformation time t

Fitted Frames for Single-Figure Objects

- Objective
  – Like fitted frames for boundary (Cartan), carry local geometry,
    - **But here of the closure of interior, not just the boundary**
    - Like boundary fitted frames, capture local curvatures,



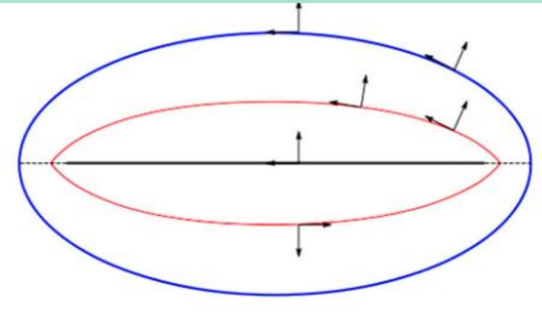
  – Provides a coordinate system for inter-point geometry
    - Rotations: curvature
    - Inter-point vectors

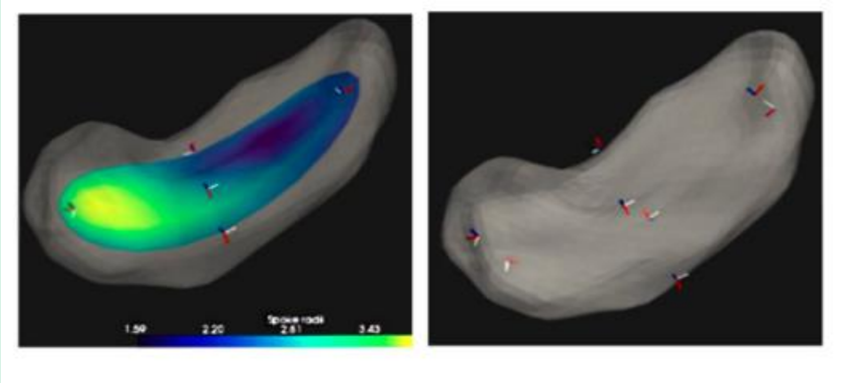


  – Do it with good correspondence across the object population
- Avoids need for alignment
  – No dependence on alignment scale

Fitted Frames for Ellipsoid, 3

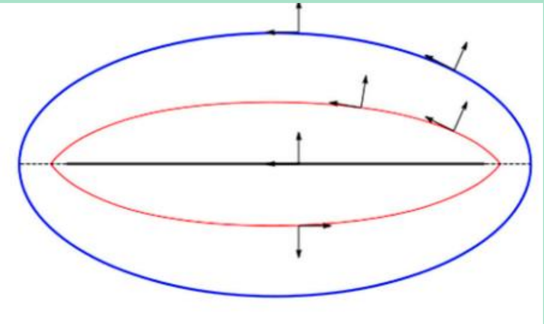
- On onion skins
  - Thus on skeleton (τ_2=0)
    - Respecting side of fold, dependent on θ
    - And thus on spine ($\tau_1 = \tau_2$=0)
  - Thus on boundary (τ_2=1)
  - In 2D normal and tangent to onion skin form frame
  - In 3D
    - Third frame vector $\mathbf{f}^3$ is normal to onion skin
    - Second frame vector $\mathbf{f}^2$ is along fixed τ_1 as θ varies
    - $\mathbf{f}^1 = \mathbf{f}^2 \times \mathbf{f}^3$



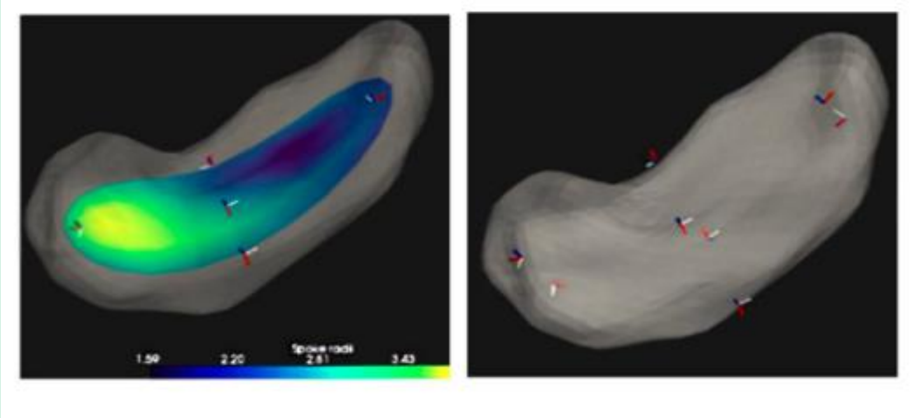


- Allows spoke interpolation
  - Rotations of frame
  - r interpolation recognizing dr properties

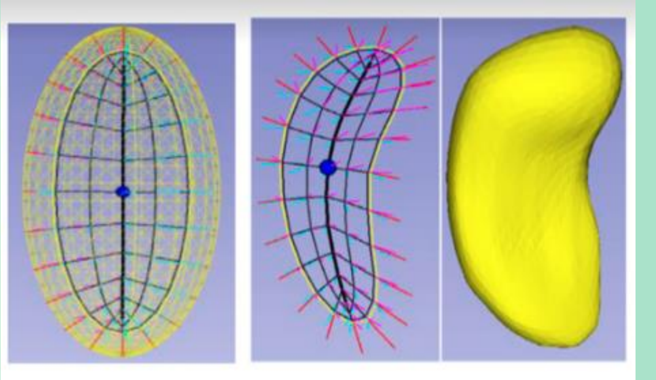
Fitted Frames for 3D, Single Figure Object of Spherical Topology



- With proper diffeomorphism, same definition as from ellipsoid
  - In 3D
    - Third frame vector $\mathbf{f}^3$ is normal to onion skin
    - Second frame vector $\mathbf{f}^2$ is along fixed τ_1 as θ varies
    - $\mathbf{f}^1 = \mathbf{f}^2 \times \mathbf{f}^3$

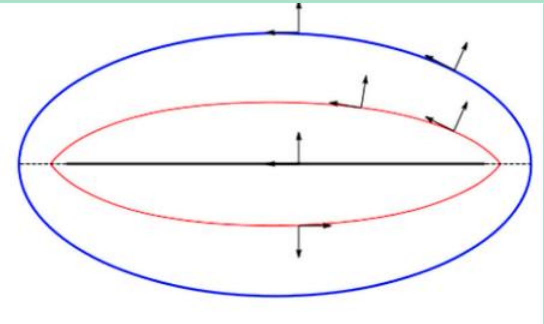


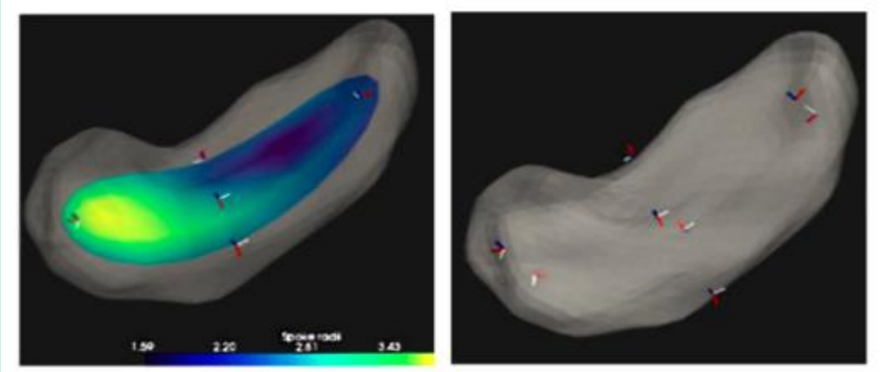
- Approximation by carrying $\mathbf{f}^1$ and $\mathbf{f}^2$ by diffeomorphism from ellipsoid

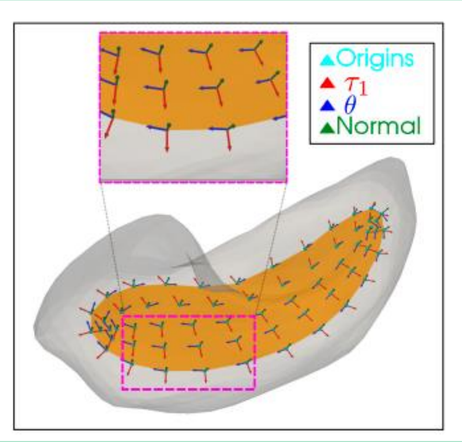


Affine Fitted Frames for 3D Single Figure Object of Spherical Topology

- Carry $\mathbf{f}^1$, $\mathbf{f}^2$, and $\mathbf{f}^3$ by diffeomorphism from ellipsoid
  - Will no longer have unit lengths
    - Lengths form features
  - Will no longer be mutually orthogonal
    - Angles form features



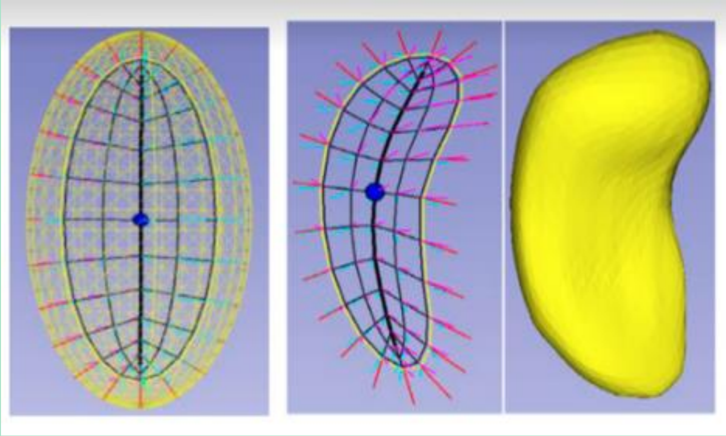


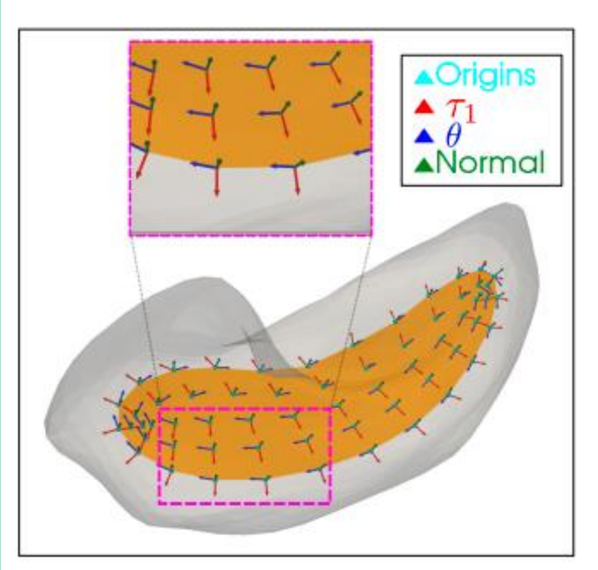


Affine fitted frames to a hippocampus skeleton [Z Liu]

Skeletal Features for Single Figure Object

- Skeletal positions
  - Ideally relative to
    - Center point frame, or
    - Neighbor skeletal position frame
- Spoke lengths
- Affine frame lengths
- Directions
  - Frame vector directions
  - Affine frame directions
  - Ideally, all relative to local frame
- For statistics, spoke directions and frame rotations are Euclideanized using PNS

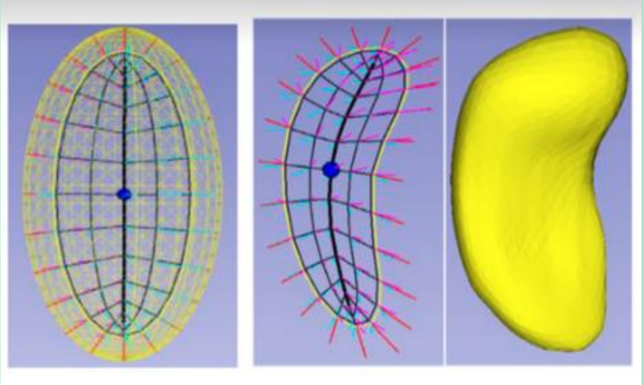




Affine fitted frames to a hippocampus skeleton [Z Liu]

Summary of Production of Skeletal Features with Correspondence

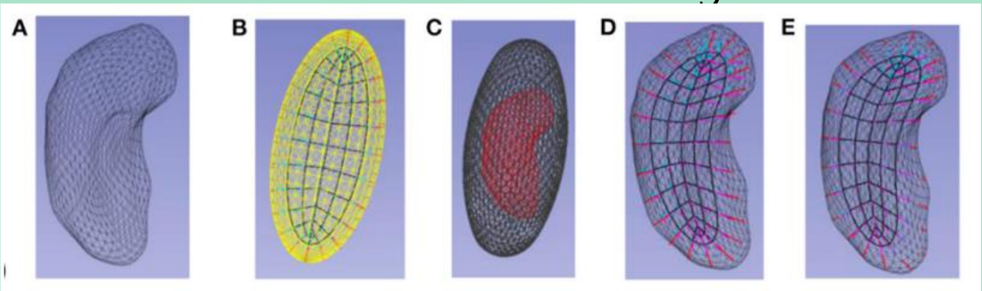
- Let rep'n of each training sample come from diffeomorphism of the same ellipsoid, recognizing
  - Vertices and crests
  - Boundaries, using CMC flow
  - Spoke loci and radial lengths
    - From skeleton to boundary
    - From spine to skeletal fold



- And producing correspondence via skeletal coordinates (θ, τ_1, τ_2)
  - Achieved by fitted frames via onions skins
    - With directions and positions measured via local frames
- Avoids alignment by use of frames fitted to onion skins

Fitting an S-rep to a Boundary Mesh

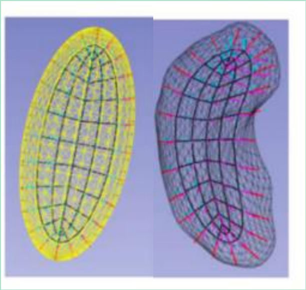
- Fitting rather than generated from boundary to fix branching topology
- Fitting to boundaries
  - Optimization [Z Liu]



    - Stage 1: approximate diffeomorphism to yield correspondence
    - Stage 2: Refinement optimization: Penalties:
      - 1) foremost, a term heavily penalizing crossing of the spokes, via κ_{r_i}
        - » Could be a hard constraint
      - 2) the deviation of the implied boundary from the target object boundary;
      - 3) the deviation of the angle of the spokes from the corresponding boundary normal
      - Could use the difference in corresponding spoke lengths
    - Code at slicersalt.org
  - Alternatives on next slides

Fitting an S-rep to a Boundary Mesh, 3

- By temporal stages producing reverse diffeomorphsisms to yield stage to stage small deformations



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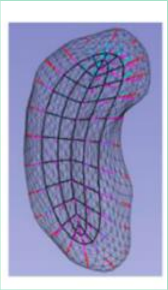
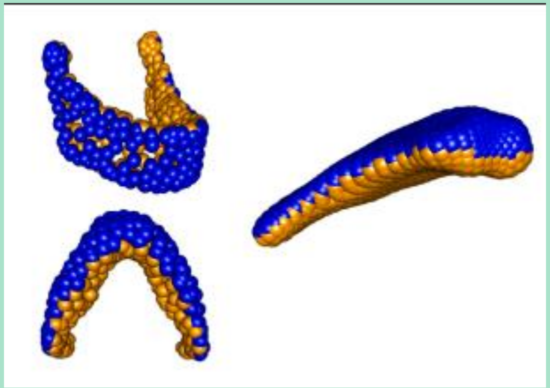
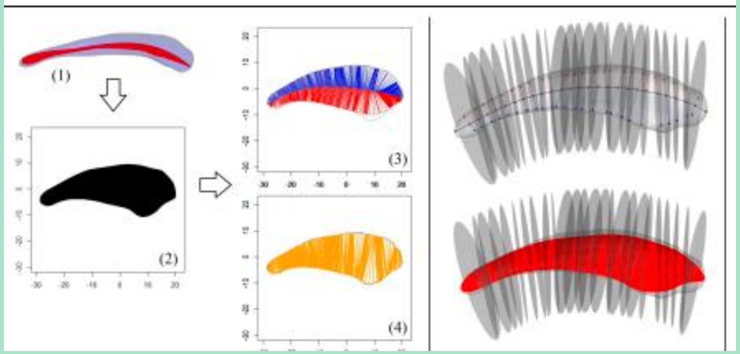


  – Each of these is an optimization: Penalties:
    - 1) foremost, a term heavily penalizing crossing of the spokes, via κ_{r_i}
      – Could be a hard constraint
    - 2) the deviation of the implied boundaries between stages
    - 3) the deviation of the angle of the spokes from the corresponding boundary normal
    - 4) the deviation of the implied crest to the later stage crest
      – Also for vertices
    - 5) the difference in corresponding spoke lengths
  – In late stages of development [Tapp-Hughes]

Taheri Computing a Plane-Sweep S-rep from a Boundary Mesh

- Find shortest spokes to boundary from each interior point
  - Find pairs with largest angles
- Classify paired spokes into top and bottom
  - Straighten and fit planes
- Compute spine
  - Requiring relative curvature criterion
- Optimize volume coverage, skeletal symmetry, and lowered curvature





Cm-reps via PDE [Yushkevich]

Biharmonic (Laplace-Beltrami)^2 operator offers an elegant solution

Solve:

$$\nabla_{\mathbf{x}}^4 R^2 = \rho$$

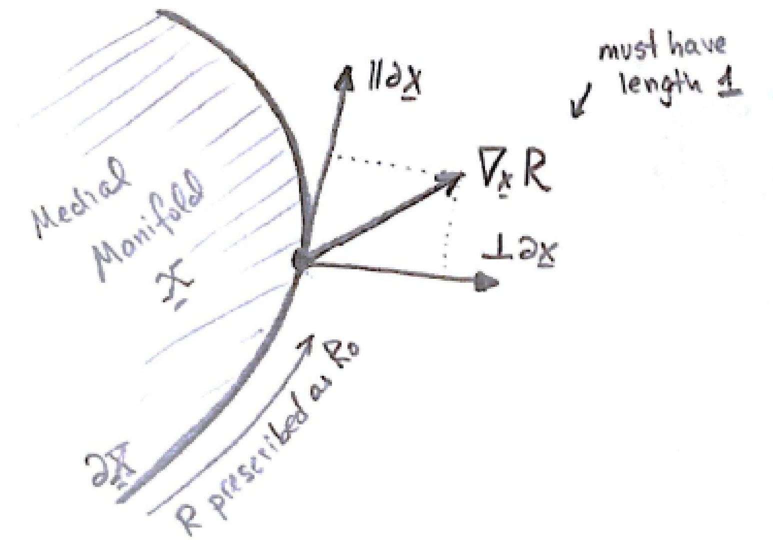
subject to:

$$\begin{cases} R = R_0 \\ D_{\perp \partial \mathbf{x}} R = \sqrt{1 - (D_{||\partial \mathbf{x}} R_0)^2} \end{cases} \text{on } \partial \mathbf{x}$$

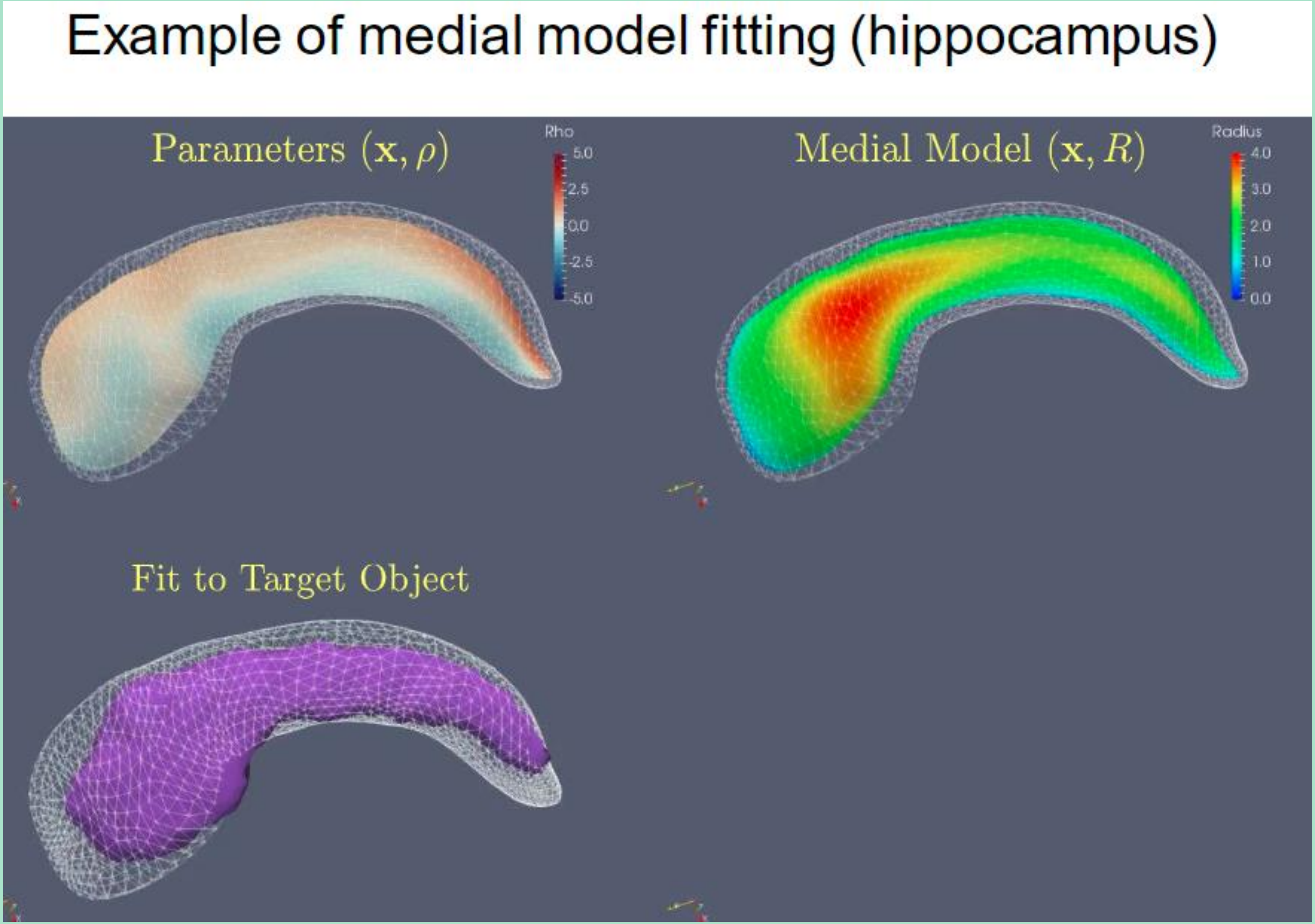
Laplace-Beltrami Operator (LBO)

$$\nabla_{\mathbf{x}}^2 f = \mathrm{div}_{\mathbf{x}} \nabla_{\mathbf{x}} f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\eta} \left(\sqrt{g}\, g^{\mu\eta} \frac{\partial f}{\partial u^\eta} \right)$$

- The 4th order PDE admits two boundary conditions
  - Dirichlet and von Neumann
- The non-linear constraint can be made linear w.r.t. the unknown function R by introducing new parameter function R_0



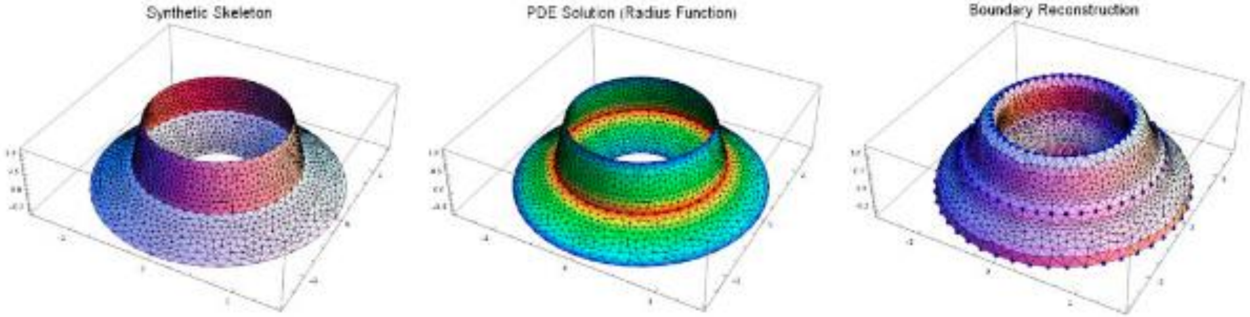
Cm-reps via PDE [Yushkevich], 2



Cm-reps via PDE [Yushkevich], 3

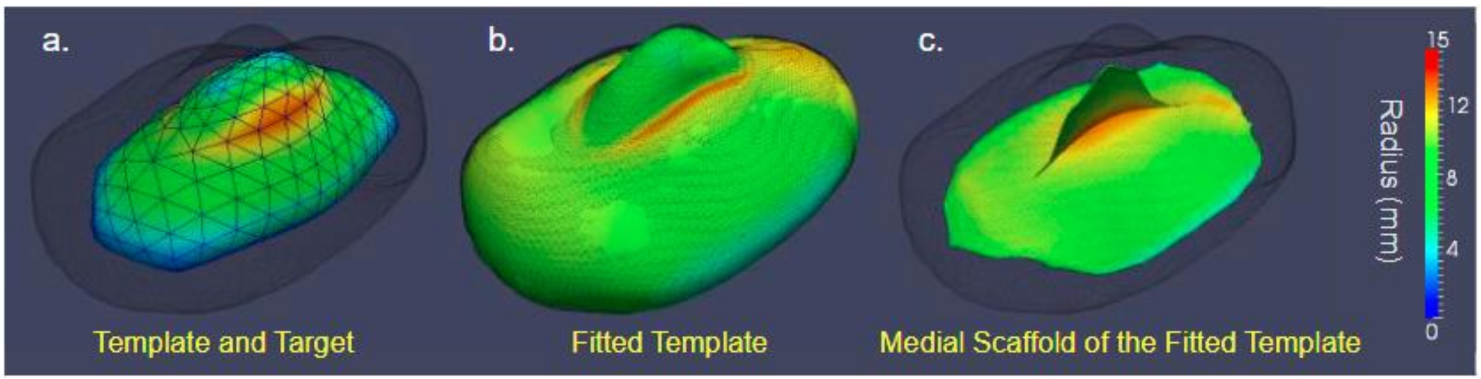
Limitations of the PDE-based approach

- Computationally expensive
  - Must solve a PDE (large sparse linear system) every iteration
  - *R*, **y** depend on ρ globally (lots of derivatives to compute)
- Does not handle tubular structures in 3D
- Limited to simple branching configurations
  - How to handle seam-edge and seam-seam intersections?

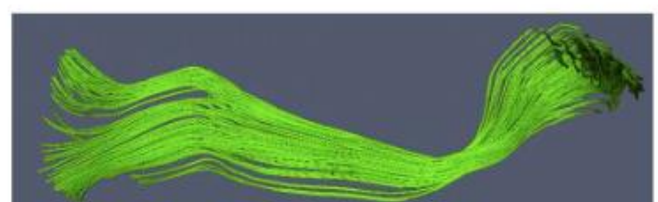


Cm-reps via Medial Linkage Preservation [Yushkevich]

The boundary-first approach, for the first time, allows deformable medial modeling of objects with **branching** Blum medial axis

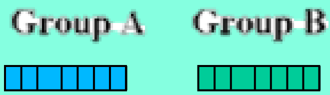


It also has the potential to model tubular and part-sheet/part-tube objects

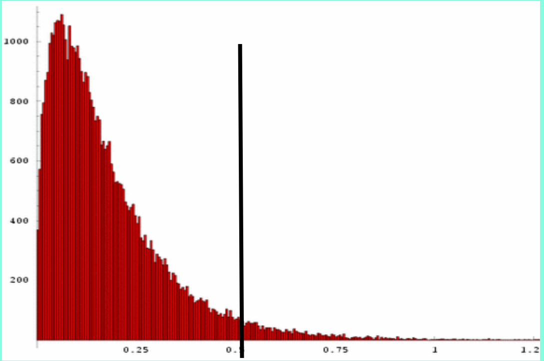


Univariate Hypothesis Testing on a Curved Surface

- Test is by **DOUBLE NEGATIVE**:
  Reject (null) hypothesis *that* the two classes are *not different*
  - I.e., reject hypothesis that two classes are the same, i.e., observed differences come from random sampling



- Typically tests on magnitude of differences between class means: m_A and m_B
  - $T_0 = d(\mu_A , \mu_B | / (s_A^2/n_A + s_B^2/n_B)^{1/2}$



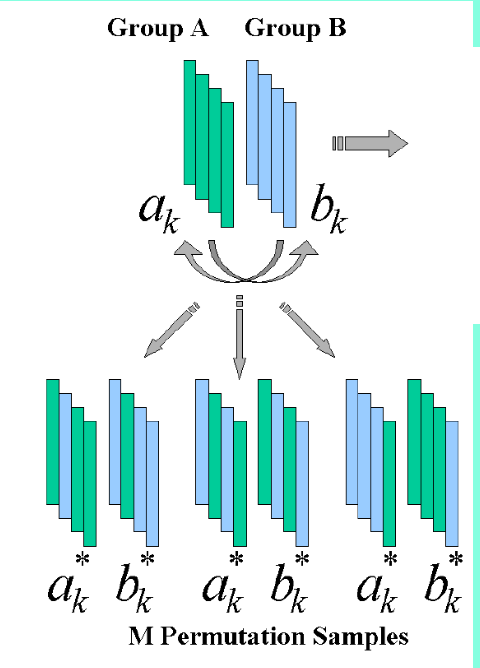
- Create distribution of T under null hypothesis empirically
  - Under null hypothesis groups are not different, any permutation produces an equivalent T (normalized)
  - Over all permutations produces empirical T distribution
  - With that distribution see percentile of T_0 in that distribution: p-value =
    #Perms larger / #Perms total

Hypothesis Testing of Shape with Locality

- **Goal:** given training data sets of objects $\underline{\mathbf{z}}^k$ in two classes, determine whether there are significant differences between the classes and if so, where
  - Where (locality): positions or other parameters
  - Training data: $\{\underline{\mathbf{z}}^k \mid 1 \leq k \leq n_{A;}\ n_A+1 \leq k \leq n_A + n_B\}$
- **Method:** Hypothesis test with initial Geometric Object Property(ies) (GOPs) at each location
  - A GOP may be a tuple, e.g., object normal direction
  - With corrections for multiple comparisons, which will lead to a different threshold for each location × GOP
    - Commensuration by turning T values into p values
    - Making p values for each feature std Gaussian
    - Then **decorrelation** via PCA

Permutation Tests on >1 Variable μ-diff fixup

- Commensurate by transforming each mean difference into a probability via its histogram
- Make distributions same and joint distribution Gaussian by turning each distribution into a standard Gaussian
  - Two cumulative histogram transformations: quantiles have uniform probability
    - Cumulative dist. of v'ble → uniform
    - Quantile f'n of normal → st'd Gaussian
- What's left is handling covariance of transformed variables



Decorrelating the Standard Normal Variables

- Note that all hyperplanar (including dimension 1) cross-sections through the mean of a Gaussian are Gaussian
  - Also, the principal dimension-1 cross-sections yield uncorrelated variables
- Σ_U is made up or correlated standard 1-dimensional Gaussians
  - So the cross-sections are not principal (nor orthogonal)
- Use PCA on Σ_U to produce new, uncorrelated variables formed as the eigenvector directions
  - Multiple test correction assuming non-correlation is applicable
  - For shapes (U formed from GOPs) there will be few such variables (ones with low eigenvalues can be cut out)

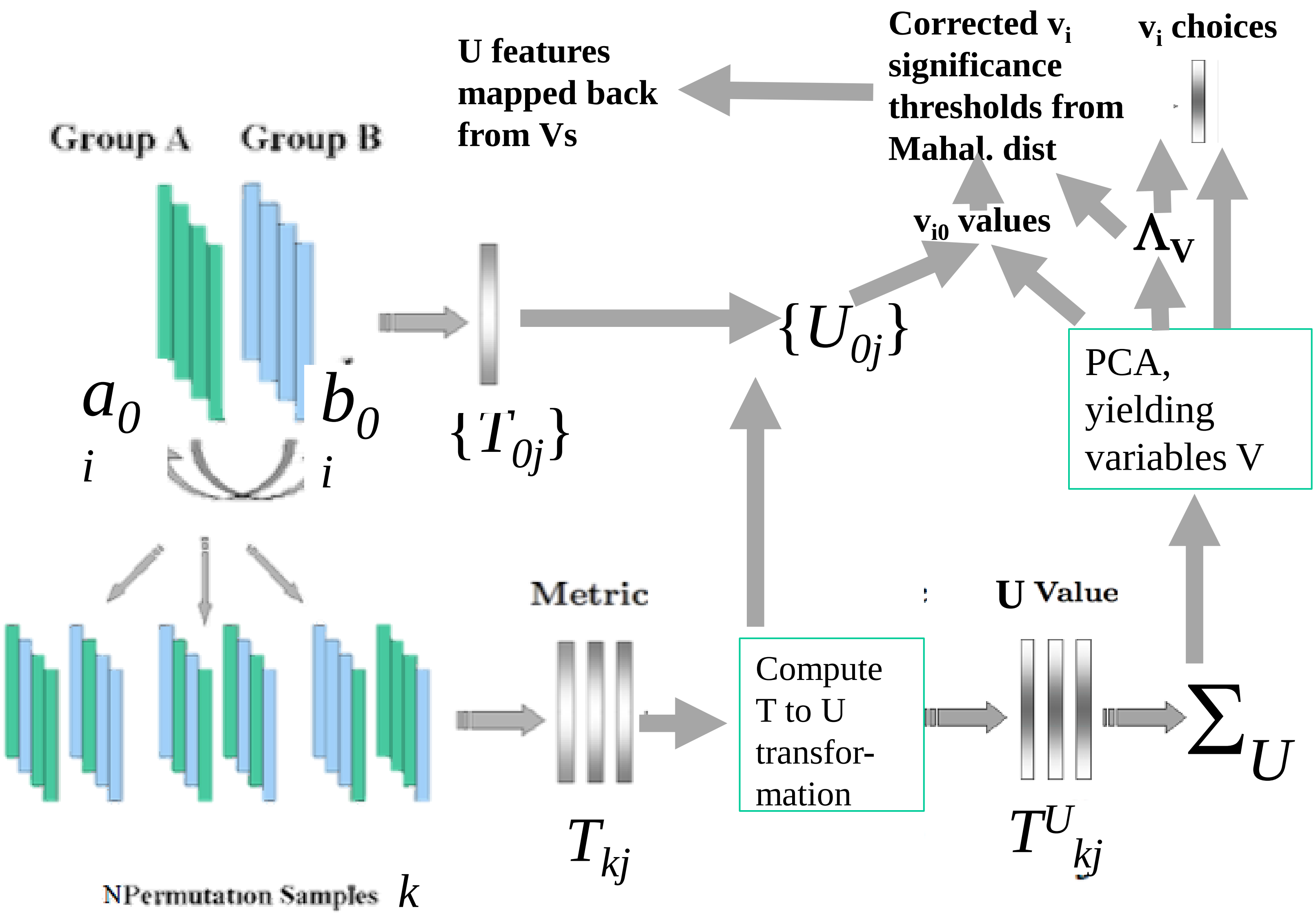
P-value correction via FDR or FWER

- False Discovery Rate (FDR)
  - More relaxed assumptions
    - More power than Bonferroni, higher specificity than uncorrected
  - Used in fMRI, VBM and Deformation field analysis
- FDR: Proportion of false positive tests among those test for which H_0 is rejected
  - Bounds expected rate among those tests that show significance only.
- FWER correction: Rate of false positives among all tests, whether or not H_0 is rejected

1. Controlling False Discovery Rate: A practical and powerful approach to multiple testing, Y Benjamini, Y Hochberg, J.R. Stat Soc Ser B 57 1995
2. Thresholding of Statistical Maps in Functional Neuroimaging Using the False Discovery Rate, CR Genovese, NA Lazar, T Nichols, NeuroImage 15 2002

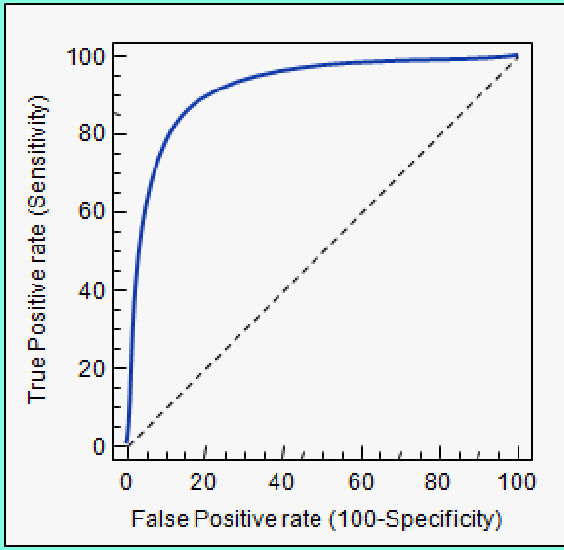
Slide: M. Styner

Corrected Multivariate Hypothesis Testing: by Permutation Tests [Pizer]



ROC Analysis

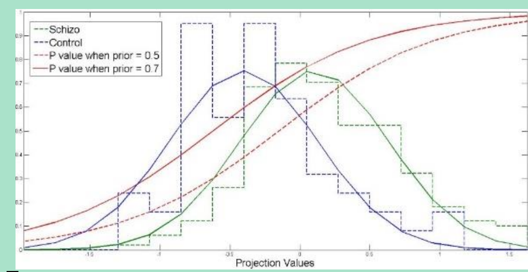
- An important means of evaluating any medical procedure involving a detection of a signal
  - Signal can be classification of voxel as the object to be segmented



- True positive rate (*sensitivity*) plotted against false positive rate (1-*specificity*)
  - TPR: P(correct decision | signal present)
  - FPR: P(wrong decision | signal absent); a measure of conservatism
  - Common measure: area under ROC (AUC)
    - Equivalent to 2-alternative forced-choice fraction correct

AUC by Random Holdouts

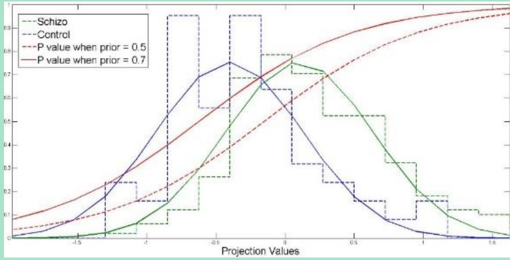
- Assume data has N_i in class i
- Pick a subdivision fraction:
  f = # training cases / N = # all cases, e.g., 80%
  - # test cases is complement
  - In each holdout # of test cases is (1-f) N_1 in class 1 and (1-f) N_2 in class 2, with the cases chosen at random
- Run the experiment many times
  - Each test case yields a d value, leading to a class choice, given a threshold



  - Yield TPR and FPR for various thresholds
- Two options for combining trials
  - Average AUCs per random set
  - Combine FPR,TPR data over sets, to yield a single AUC

Combining random holdouts results into an AUC

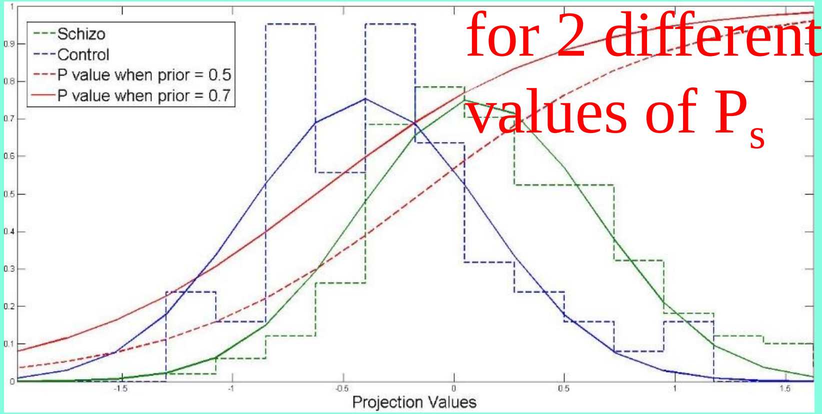
- Each test case yields a d value along separation direction
- But d values are not commensurate across random holdouts
  - They have different separation directions
- So turn d values into p values via Bayesian analysis
  - Then the set of p values for each class can be coalesced to produce 2 histograms, which can yield an ROC and thus an AUC



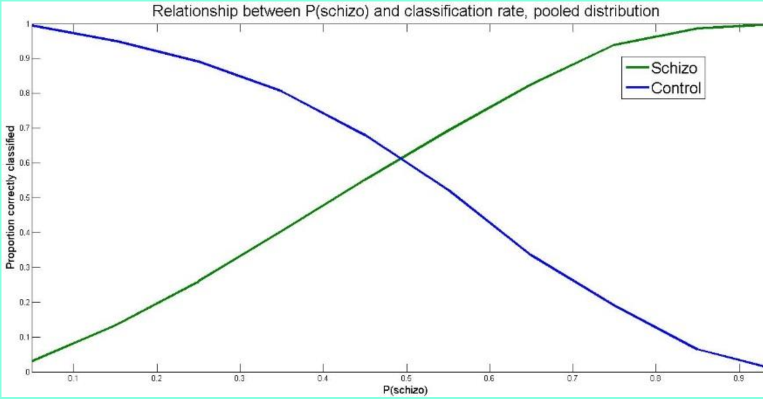
Going from histograms on d to P(schizo | D) function

- Bayesian formulation
  - Two histograms are Gaussians with common variance from histo's P(d | schizo) and P(d | control)
    - $P(\, schizo|\, d) = \dfrac{p_s \, p(d|schizo)}{p_s \, p(d|schizo) + \, (1 - p_s)\, p(d|control)}$
  - There is a parameter, P_s, the prior probability of being schizo
    - Each value of P_s yields a different P(schizo | d) function
    - Applied to test data, each value of P_s yields a different true positive rate and true negative rate
    - These rate curves yield an ROC

P(schizo|D) for 2 different values of P_s



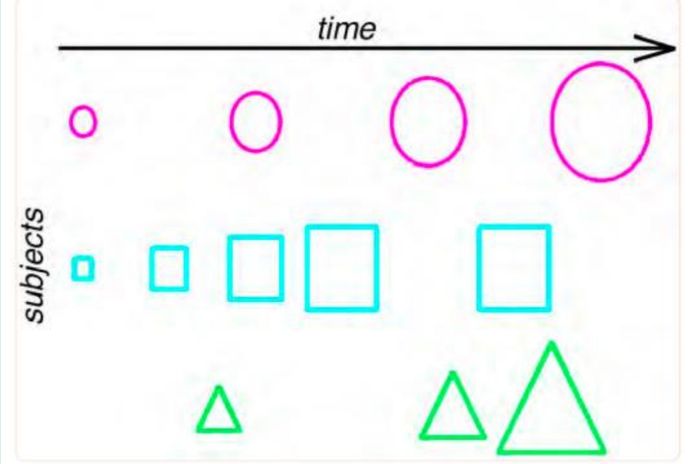
d



Longitudinal Analysis of Shape

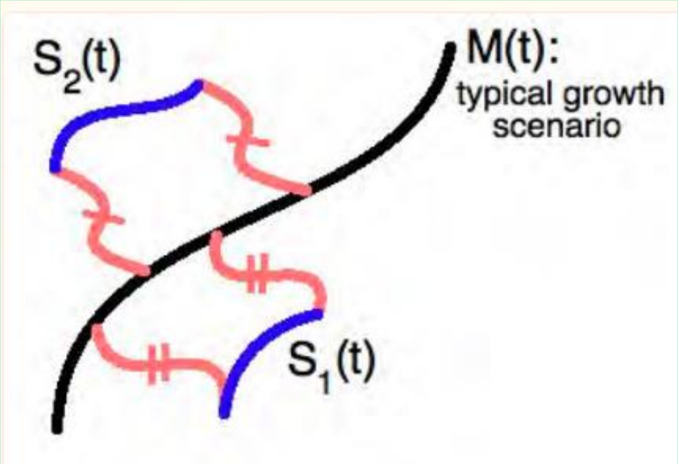
- How does shape change over time?
- Longitudinal data set:
  - Set of homologous objects (e.g., anatomical structures), each object being observed repeatedly at several time points; for now non-cyclic, e.g., aging
  - Shape varies across individuals
  - For each individual shape temporal paths differ
    - Temporal sampling may differ among individuals
    - Initial times and rates of changes of shape longitudinal changes may differ among individuals
- Two approaches
  - Use object features $\underline{\mathbf{z}}$ on curved manifold and study $\underline{\mathbf{z}}(t)$
    - See Fletcher lectures in this course and his papers
    - Schiratti, …, Durrleman, for mixed effects models on manifolds: “Learning spatiotemporal trajectories from manifold-valued long’l data”
  - Study deformations in time: $\phi(\underline{x},t)$. See upcoming slides

Longitudinal Analysis of Shape via Space Deformations

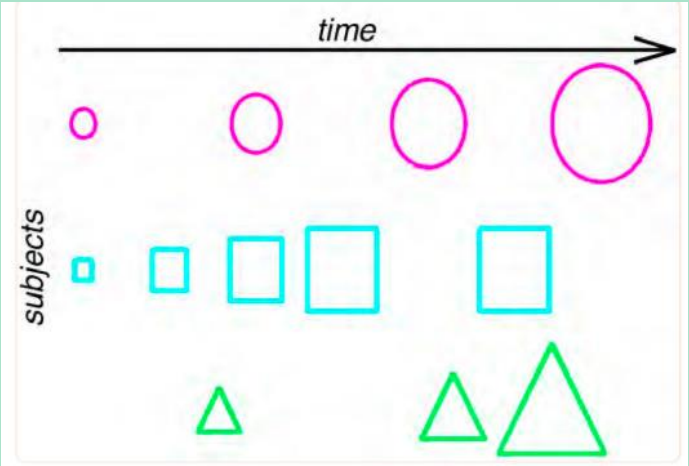


- Data
- Issues
  - Need to do subject by subject deformations across time:
  - Need to do subject by subject temporal deformations from atlas
    - Time passes on differently for different subjects
  - Need to do inter-subject deformations from atlas
    - Thus need an atlas, M(t), typically a kind of mean
  - Want to include covariates such as age (especially) and gender in the analysis

Longitudinal Analysis of Shape via Space Deformations, 3

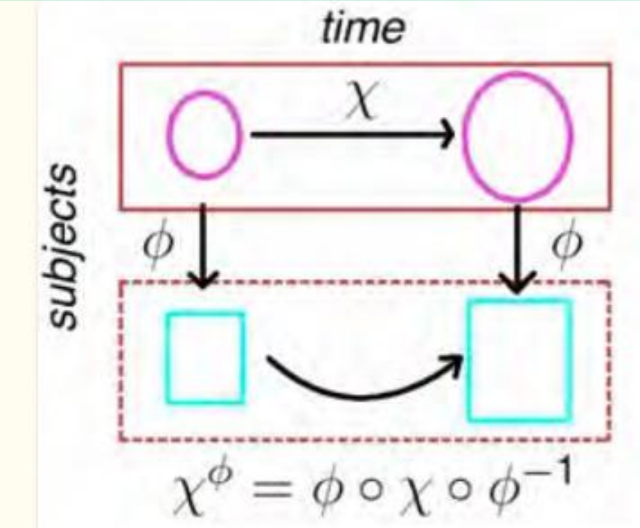


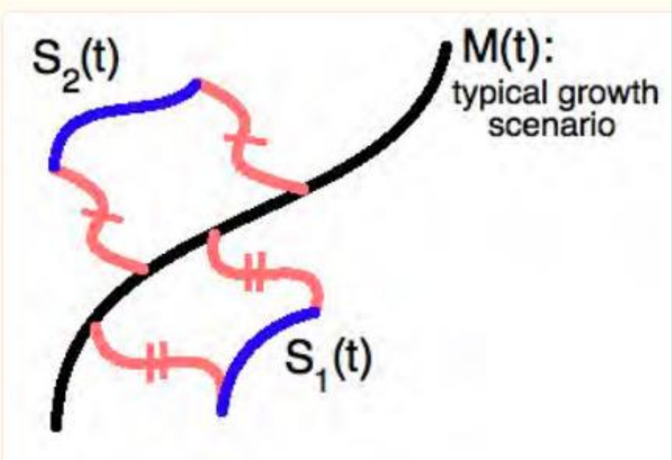
- Issues
  - Need to do subject by subject deformations across time: $S_i(t)$
  - Need to do subject by subject temporal deformations from atlas
    - Time passes on differently for different subjects
  - Need to do inter-subject deformations from atlas
    - Thus need an atlas, M(t), typically a kind of mean



Subject-specific Longitudinal Analysis of Shape via Space Deformations

- Let $\chi(t)$ be deformation of shape across time of atlas
  - Let $\chi_j(t)$ be deformation of shape across time of subject j
  - Let $\phi_{jt}(\underline{x})$ be deformation of shape at time t from the atlas of subject j





- Via LDDMM approach minimize

$$E(\chi) = \sum_{t_i} \mathrm{d}(\chi_{t_i}(M_0), S_i)^2 + \gamma^{\chi}\mathrm{Reg}(\chi)$$

Subject-specific Longitudinal Analysis of Shape via Space Deformations

- Let $\chi(t)$ be deformation of shape across time of atlas
  - Let $\chi_j(t)$ be deformation of shape across time of subject j
- But time runs differently for each subject
  - Let $\psi(t)$ deform time:

$$\Phi(x, y, z, t) = (\phi(x, y, z, \psi(t)), \psi(t))$$

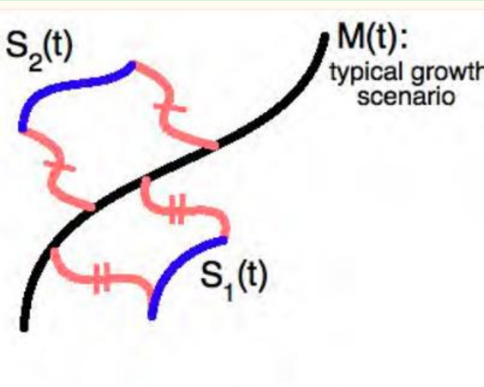
where the geometrical part has the form

$$\phi(x, y, z, t) = \chi_t \bigcirc \phi_0 \bigcirc \chi_t^{-1}(x, y, z)$$

- Via LDDMM approach minimize

$$E(\phi, \psi) = \sum_{t_j} \mathrm{d}(\phi(S(\psi(t_j))), U_{t_j})^2 + \gamma^{\phi}\mathrm{Reg}(\phi) + \gamma^{\psi}\mathrm{Reg}(\psi)$$

  - where U_{tj} is the shape of object j at time t



subject-specific approach