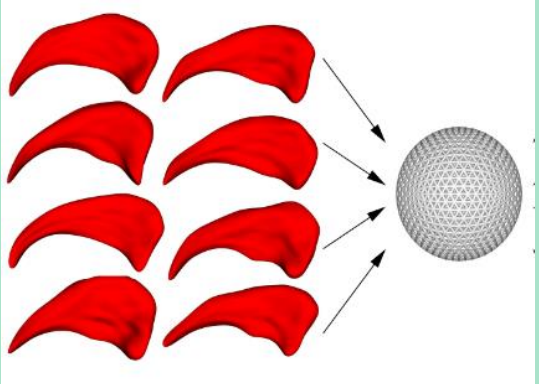
Non-PDM Representations and Statistics

- Srivastava boundary geometry modulo corresp.
- Skeletal representation
  - Medial and skeletal mathematics
  - S-reps
    - Skeletal points, spokes
    - Diffeomorphisms from ellipsoids, fitted frames
    - Taheri via swept planar cross-sections
    - Fitting to boundaries
      - Optimization
      - CNN
  - Cm-reps [2 lectures by P. Yushkevich]
    - Based on PDE
    - Based on splines in x and half-width r
    - Fitting to boundaries
- Statistics on Riemannian manifolds [Fletcher]

Srivastava boundary geometry modulo correspondence!

- Idea is to mod out not only alignments but also correspondence along the boundary
  - Base parameterization: $\underline{x}(\underline{u})$; reparam'n: $\underline{x}(\underline{\gamma}(\underline{u}))$
    - In 2D $u = \theta$, parametrizing circle
    - In 3D $\underline{u} = (\theta,\phi)$ parameterizing sphere
  - Want representation independent of boundary parameterization



- References
  - Curves: Book: A Srivastava & EP Klassen. *Functional and Shape Data Analysis*. Springer, 2016
    - Be in control of Plane Curves chapter

Srivastava boundary geometry modulo correspondence, 2

- Want representation independent of boundary parameterization
  - So inter-object distances are between equivalence classes over alignment in R^n and reparameterization
  - Thus we want the distance between two curves where one is the reparameterization or rotation of the other to be 0
    - Srivastava and Klassen show this implies that the representation of curve q parameterized by γ (denoted by (q,γ)) should be $(q \circ \gamma)/\sqrt{\dot{\gamma}}$
      - See next slide

Srivastava boundary reparametization representation

- Copied from Srivastava and Klassen

Shapes of Planar Curves

Let Γ_I denote the set of all orientation-preserving diffeomorphisms [0, 1] → [0, 1], as introduced in Sect. 4.3.3 ("orientation-preserving" means the diffeomorphism preserves direction, i.e., 0 maps to 0 and 1 maps to 1.) As described in Sect. 5.2, the re-parameterization of a curve $\beta : [0,1] \to \mathbb{R}^2$ by a $\gamma \in \Gamma_I$ is given by $\beta \circ \gamma$. In terms of the SRVFs, what is the effect of re-parameterization? Similar to the case of SRSFs in Chap. 4, it is given by the following group action. Define: $\mathcal{C}_2 \times \Gamma_I \to \mathcal{C}_2$ by

$$(q, \gamma) \mapsto (q \circ \gamma)\sqrt{\dot{\gamma}}\,. \tag{5.19}$$

The reason this is the correct action of Γ_I is as follows. Suppose we are given two curves β_1 and β_2, where each $\beta_i : [0,1] \to \mathbb{R}^2$. Assume they are related by an element $\gamma \in \Gamma_I$, so $\beta_2 = \beta_1 \circ \gamma$. If q_1 and q_2 are the corresponding representative functions, what is the relation between q_1 and q_2? First, recall that $q_1(t) = \dot{\beta}_1(t)/\sqrt{|\dot{\beta}_1(t)|}$. Likewise, $q_2(t) = \dot{\beta}_2(t)/\sqrt{|\dot{\beta}_2(t)|}$. Using the chain rule, we then compute that

$$\begin{aligned} q_2(t) &= \frac{\dot{\beta}_1(\gamma(t))\dot{\gamma}(t)}{\sqrt{|\dot{\beta}_1(\gamma(t))\dot{\gamma}(t)|}} = \frac{\dot{\beta}_1(\gamma(t))\dot{\gamma}(t)}{\sqrt{|\dot{\beta}_1(\gamma(t))|}\sqrt{\dot{\gamma}(t)}} = \frac{\dot{\beta}_1(\gamma(t))}{\sqrt{|\dot{\beta}_1(\gamma(t))|}}\sqrt{\dot{\gamma}(t)} \\ &= q_1(\gamma(t))\sqrt{\dot{\gamma}(t)}. \end{aligned}$$

Thus, we have justified the action of Γ_I defined above.

Srivastava boundary geometry modulo correspondence, 3

- Want representation independent of boundary parameterization
  - So inter-object distances are between equivalence classes over alignment in R^n and reparameterization should be
  - $\|[q_1],[q_2]\| = min_{O \in SO^n, \gamma \in \Gamma}\ d_c(q_1, O(q_2 \circ \gamma))/\sqrt{\dot{\gamma}}$,
    - $\|A,B\|$ = distance^2 between A and B
    - $[q_i]$ = equivalence class of Γ = rep'ns q_i that are reparameterizations γ of boundary of object i
    - d_C is L^2 norm on boundary representation (normal)
    - *O* is orbit over reparameterizations

Srivastava boundary geometry modulo correspondence, 3

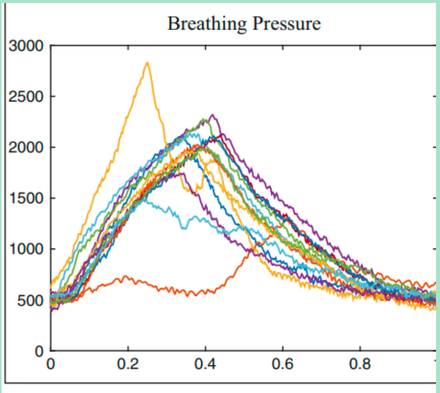
- How is geometry involved?
  - With arclength, s, parameterization, $\dot{\gamma}$ = the curvature κ(s), i.e., it involves derivatives at the curve points
  - So distance between 2 equivalence classes of curves:
  - $\|[q_1],[q_2]\| = min_{O \in SO^n, \gamma \in \Gamma}\ d_c(q_1, O(q_2 \circ \gamma))/\sqrt{\kappa}$
  -
- κ(s) on boundary is used in its representation
  - x variation (1st derivative wrt arclength) along boundary yields tangents thus normal, and their derivatives wrt arclength yield curvature

Srivastava Geodesics Between Boundaries

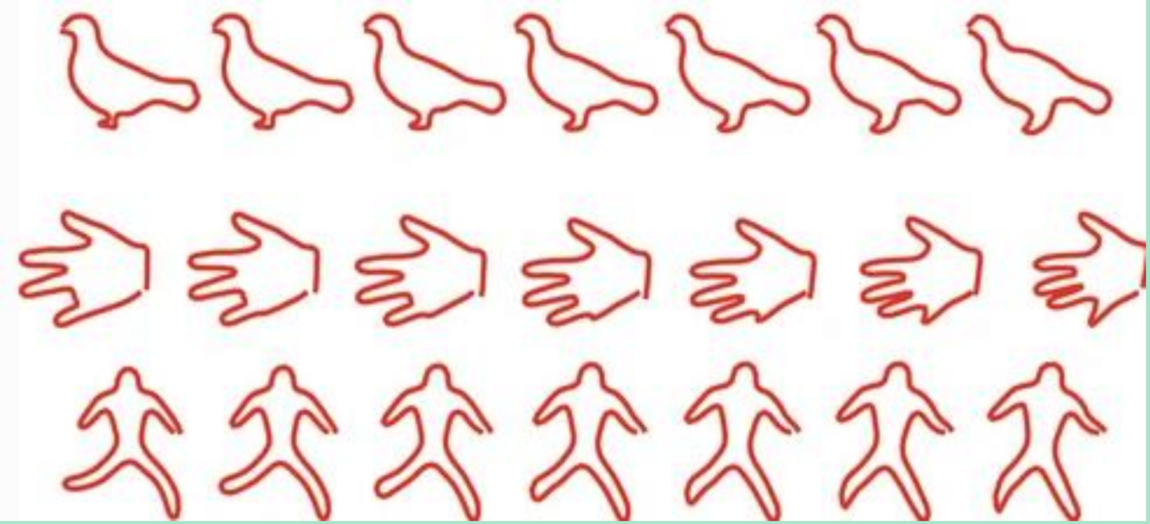
- Geodesics are according to the specified distances:

$$\|[q_1],[q_2]\| = min_{O \in SO^n, \gamma \in \Gamma}\ d_c(q_1, O(q_2 \circ \gamma)) / \sqrt{\kappa}$$

  - As are statistics: means, covariances



- Examples of geodesics paths (from S & K):



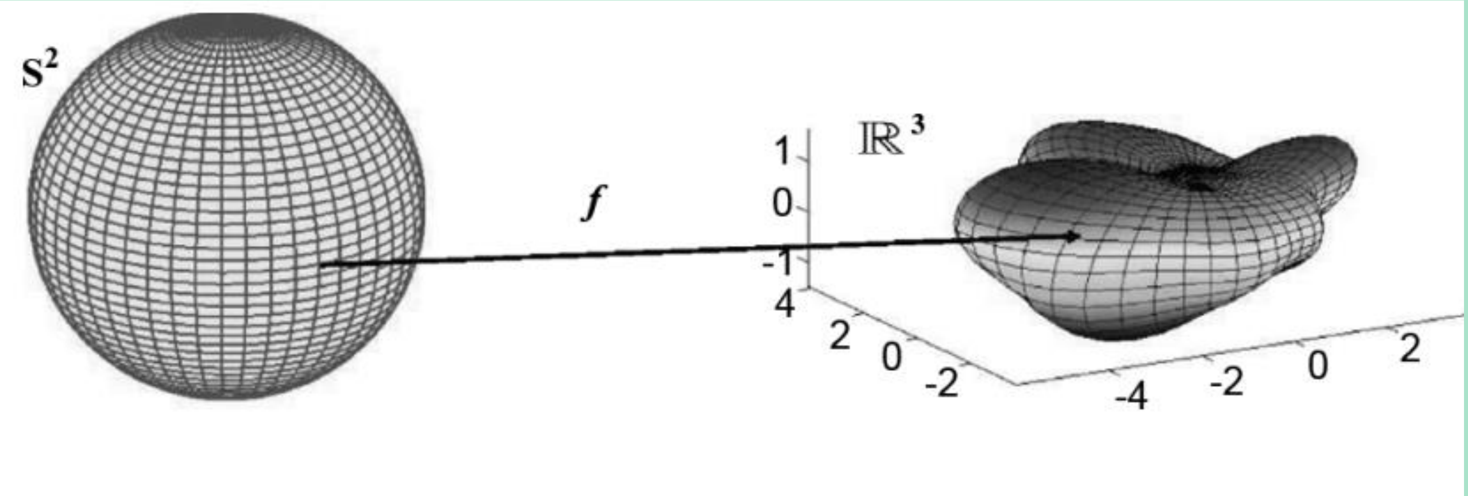
Kurtek 2D boundary geometry in 3D modulo correspondence

- References

  - Kurtek, S., Klassen, E., Gore, J. C., Ding, Z., & Srivastava, A. (2011). Elastic geodesic paths in shape space of parameterized surfaces. *IEEE transactions on pattern analysis and machine intelligence*, *34*(9), 1717-1730.
  - Jermyn, I. H., Kurtek, S., Klassen, E., & Srivastava, A. (2012, October). Elastic shape matching of parameterized surfaces using square root normal fields. In *European conference on computer vision* (pp. 804-817). Springer, Berlin, Heidelberg.
  - **Kurtek, S., Klassen, E., Ding, Z., Jacobson, S. W., Jacobson, J. L., Avison, M. J., & Srivastava, A. (2011). Parameterization-invariant shape comparisons of anatomical surfaces. *IEEE Transactions on Medical Imaging*, *30*(3), 849-858.**

Kurtek 2D boundary geometry in 3D modulo correspondence, 2

- Initial map object from sphere



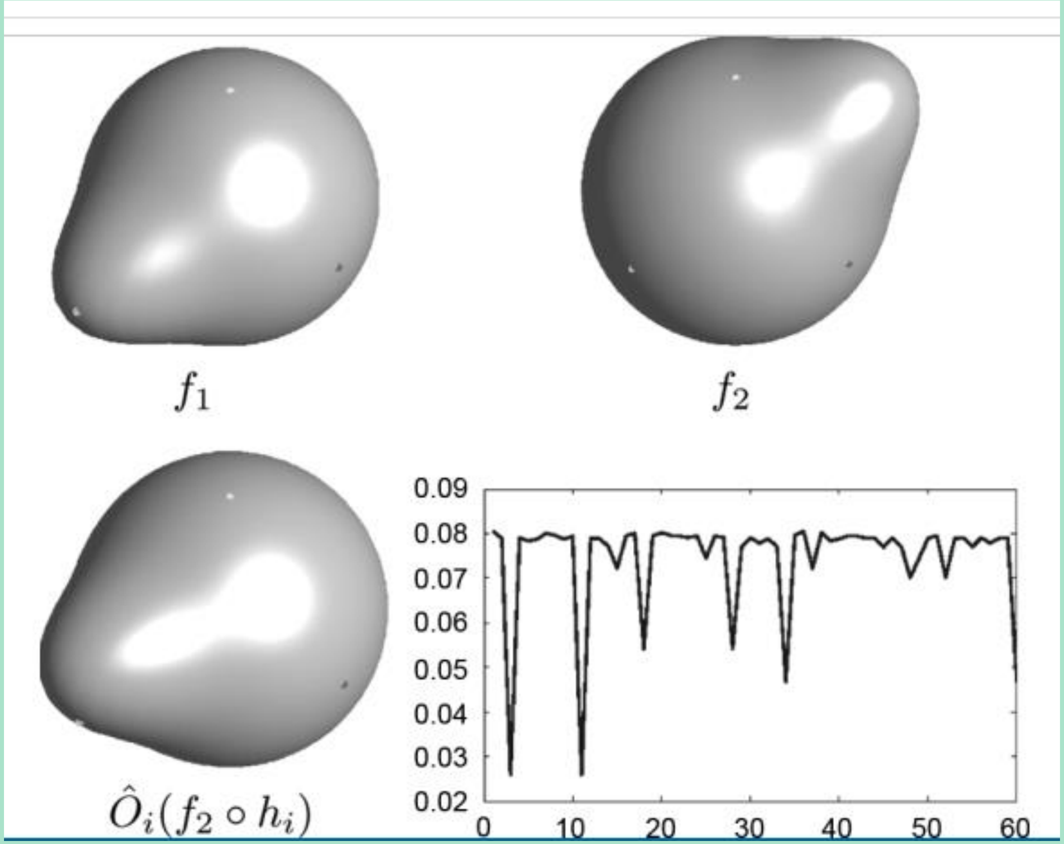
  – From Kurtek et al., TMI 2011

Kurtek 2D boundary geometry in 3D modulo correspondence, 3

- Want representation independent of boundary parameterization
  - So inter-object distances are between equivalence classes over alignment in R^n and reparameterization
  - $d([q_1],[q_2]) = \min_{O \in SO(3), \gamma \in \Gamma} || q_1 - O_\gamma q_2 ||$
    - d(A,B) = distance between A and B
    - $[q_i]$ = equivalence class of Γ = rep'ns q_i that are reparameterizations γ of boundary of object i
    - O_γ is orbit over reparameterizations
    - In q_1 and q_2 scaled versions of the objects are used
    - Minimization of the reparameterizations needs to use its Jacobian, which captures the geometry through its fitted frames

Kurtek 2D boundary geometry in 3D modulo correspondence, 4

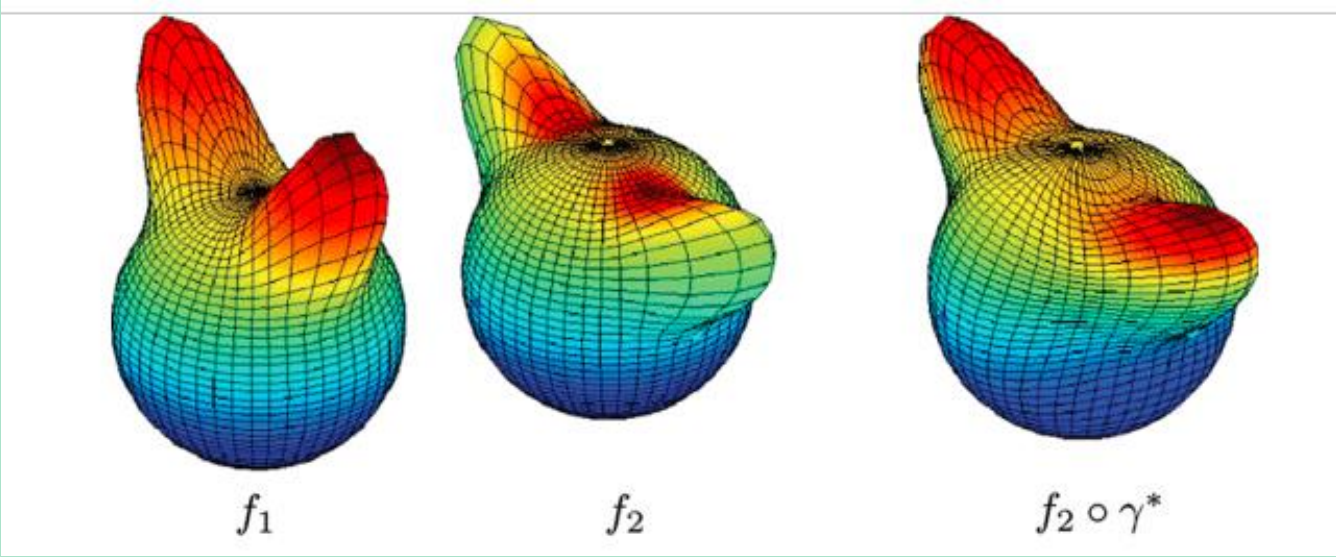
- Example 1: normalization of f_2 that minimizes distance to f_1



  - From Kurtek et al., TMI 2011

Kurtek 2D boundary geometry in 3D modulo correspondence, 5

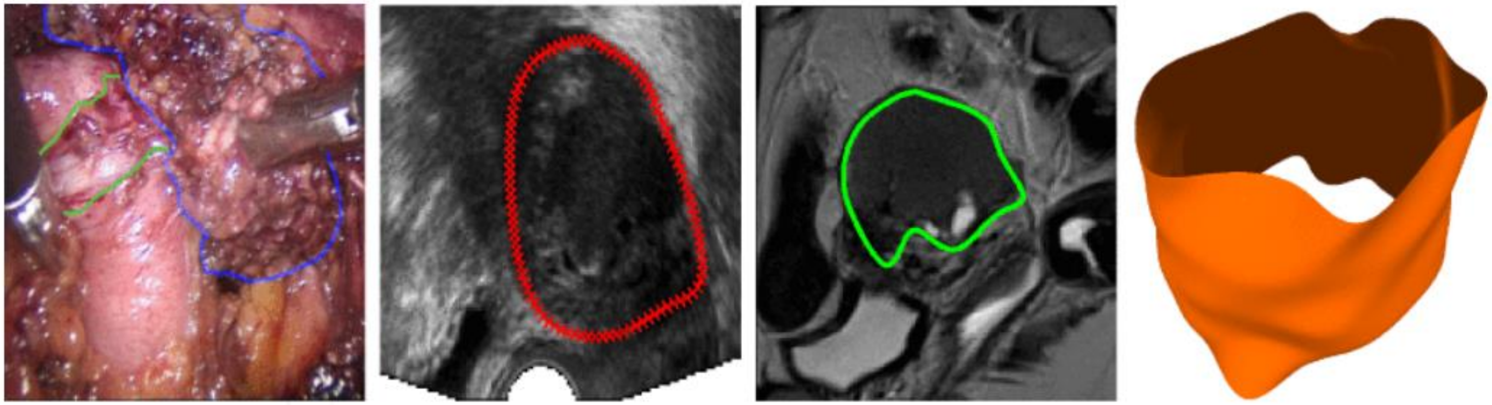
- Example 2: normalization of f_2 that minimizes distance to f_1



  - From Kurtek et al., TMI 2011
  - Distance metric allows statistics, e.g., mean

Kurtek 2D boundary geometry in 3D modulo correspondence, 6

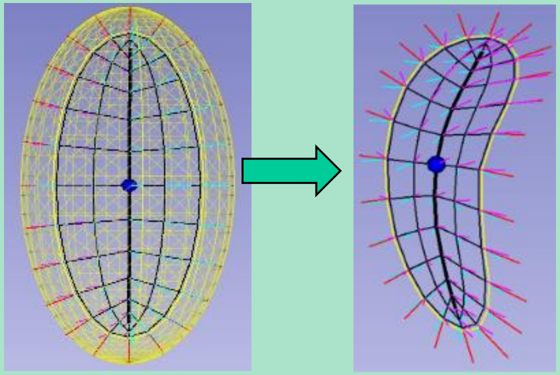
- Also has been applied to generalized cylindrical structures



(a) (b) (c) (d)

  - a) laparacopic; b) TVUS; c) MRI; d) reconstruction
  - From Samir, Kurtek, Srivastava, Canis, TMI 2014

Skeletal Representations

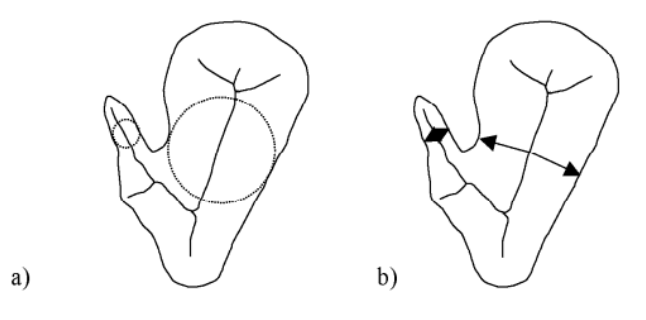


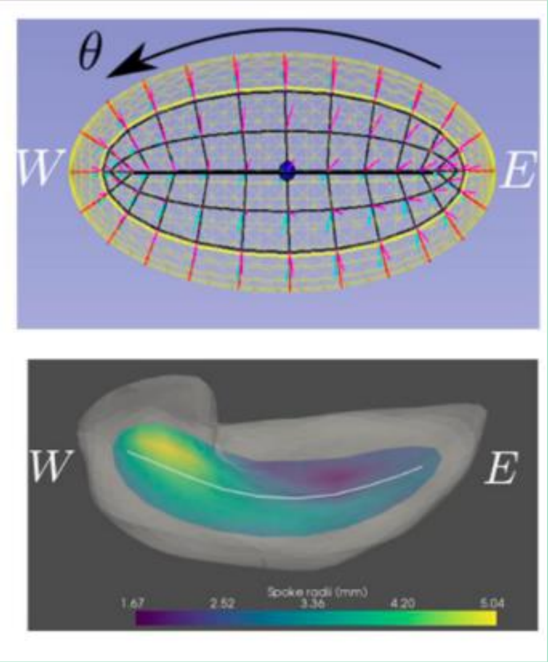
- Conceptually, skeletal representations have the following major advantages over other representations:
  - Capture object width
  - Capture curvature and direction of the object *interior*
  - Ideally, capture division of object into a tree of protrusions and indentations
  - Separate width and bending features

- Assigned readings:
  - [Siddiqi & Pizer, Ch 3: Damon; read through sec. 3.3.2
  - Pizer Frontiers (google drive); read all]

Medial and Skeletal Mathematics

- Medial representation [Siddiqi & Pizer, Chs 2 & 3], invented by Blum
  - Skeleton (medial locus) $\underline{x}$(u,v), spoke length r(u,v)
    - Bitangent spheres entirely in object interior with centers $\underline{x}$, radii r
    - Implies spoke r**U**, where **U** gives spoke direction
      - Computed by grassfire [Siddiqi, Ch. 4 in Siddiqi & Pizer]





Medial and Skeletal Mathematics, 2

- Spoke mathematics

  S = r**U**, where **U** gives spoke direction

  – When x has 2 spokes,
    - They have the same r value, and
    - The bisector of the spokes, $\mathbf{U^0}$. is tangent to the skeleton

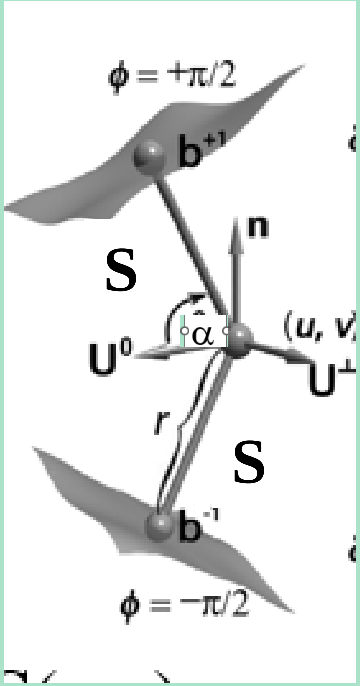
  – $\nabla_x r = -\mathbf{U^0} \cos(\text{spoke angle } \alpha \text{ from } \mathbf{U^0})$
    - $\alpha = \pi/2$ at critical points of r (max, min, and in 3D: saddle)

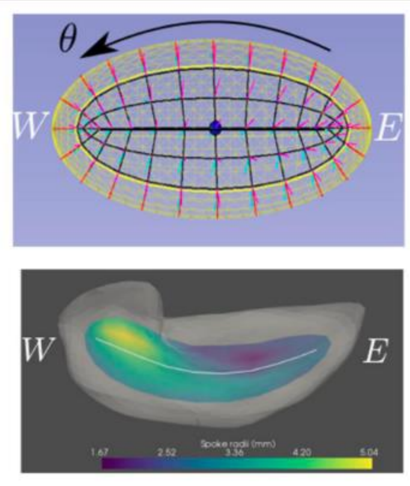
  – Each **S** is orthogonal to the boundary

  – Spoke ends b = x + r**U** imply the boundary with tolerance

- Interior and boundary positions are parameterized by figural coordinates (u,v,τ_2), with τ_2 = fraction of spoke from x(u,v)

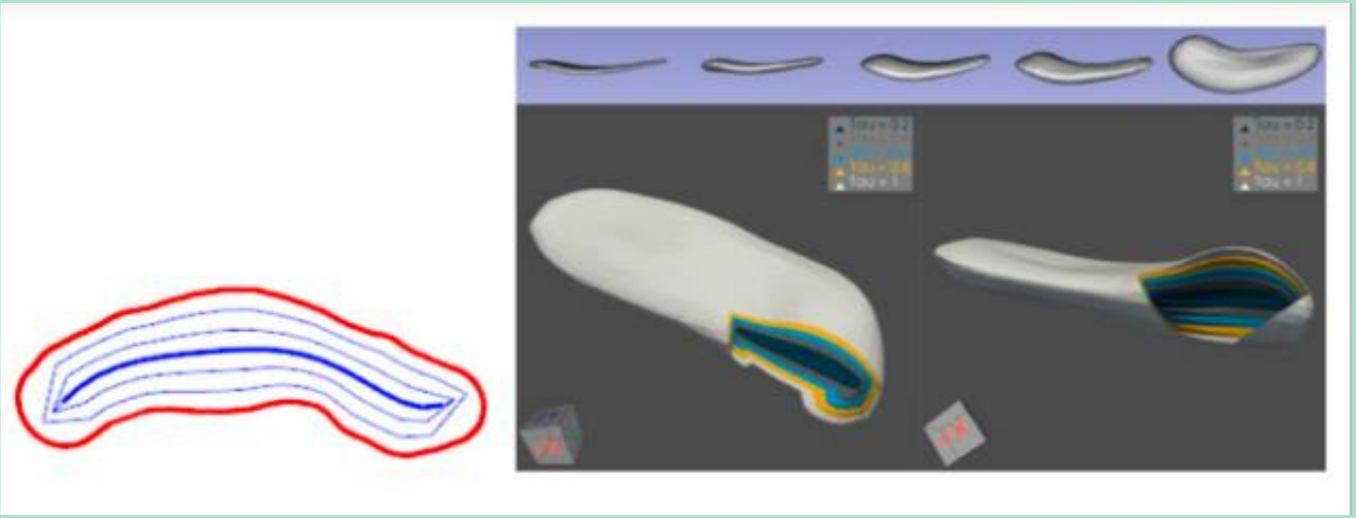
  – $\underline{b}_{\tau 2} = \psi(\underline{x},\tau_2) = \underline{x} + \tau_2 r\mathbf{U}$

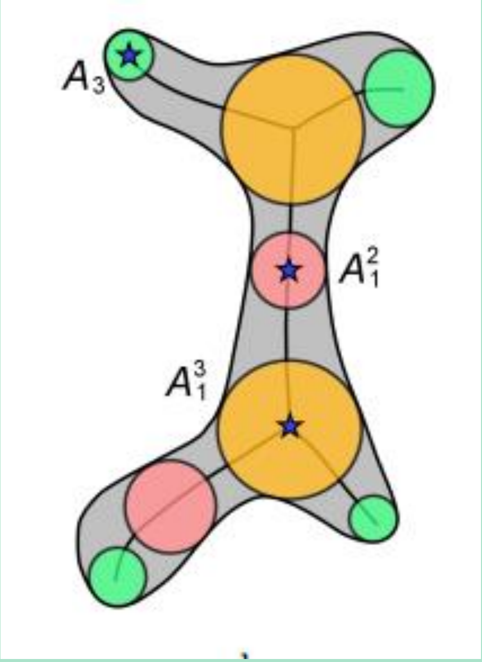




Medial and Skeletal Mathematics, 3

- Medial skeleton form
  - Commonly seen as bounded locus
  - Better thought of as folded locus with co-located pairs of x values

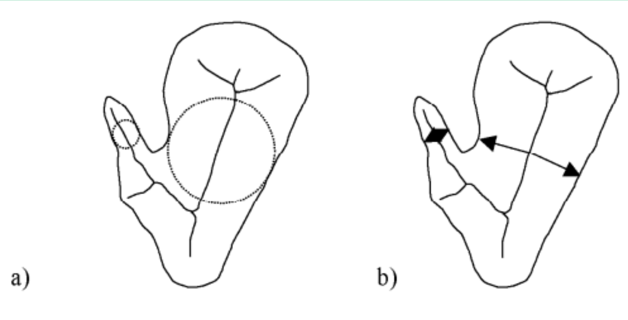


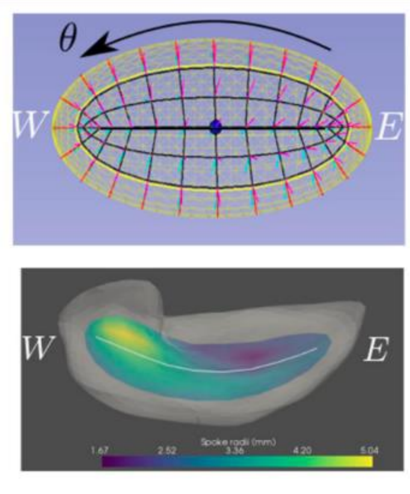


- At branch point, each side of medial locus attaches to its own side of the same side of host fold, except when boundary of branch crosses the crest of the host

Medial and Skeletal Mathematics, 3

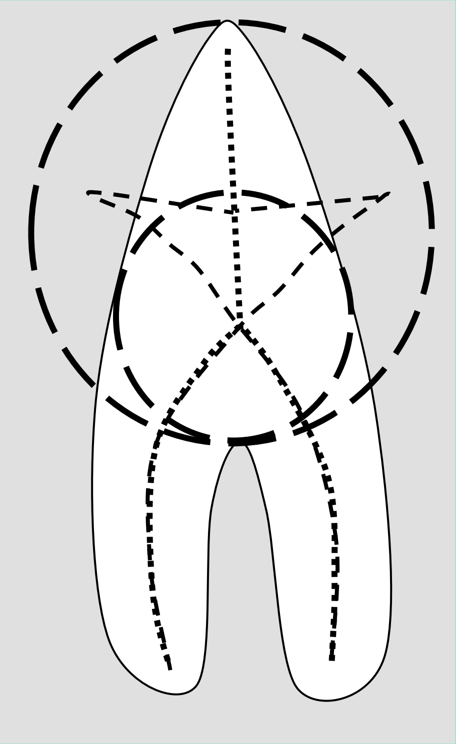
- Medial skeleton point types in 2D
  - Normal points
    - Precisely 2 points of circular tangency
    - Implies 2 points on skeleton,
      each with 1 spoke
      with equal length with the other
  - Folds where sphere osculates along crest: curvature maximizing principal direction
    - Single spoke; 3rd order (!) sphere touching boundary
    - Spoke is in limiting tangent plane to skeleton
  - Branch curves
    - Tritangent sphere tangency
    - Typically many branches due to boundary noise!





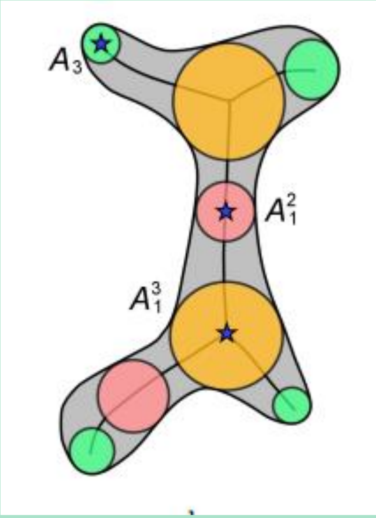
Medial Mathematics, 4

- Symmetry set: all bitangent spheres [Giblin, Ch. 2 in Siddiqi and Pizer]
- Consider the function in 2D g(u)= $(x(u)\text{-}\ x_m(u_0))^2 + (y(u)\text{-}\ y_m(u_0))^2 - r_0^2$ at $u = u_0$ with $\underline{x}$ on boundary, $\underline{x}_m$ medial
- Issue is tightness of fit: the index of the first nonzero polynomial coefficient of g(u)
  – Index of last zero coefficient gives the order
  – Its singularities produce branch and fold types
- In 3D there will be a z term, also



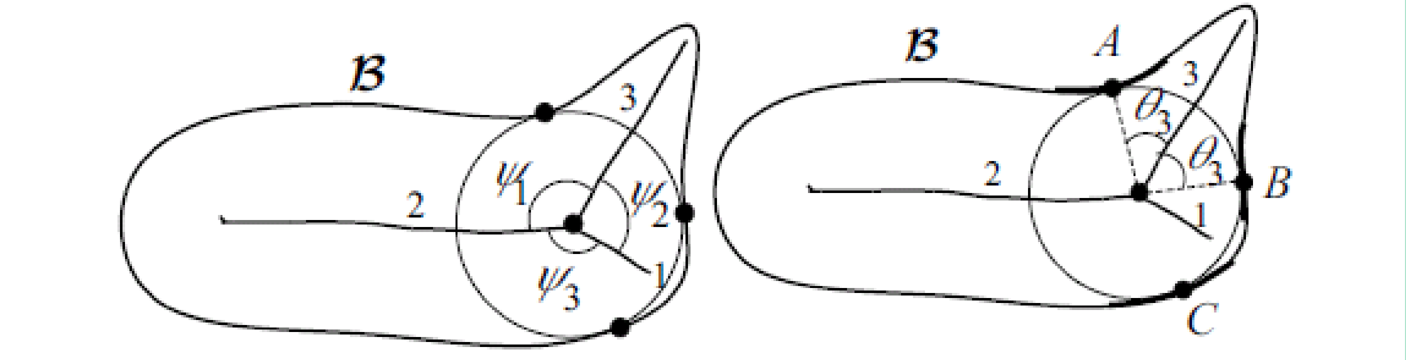
Medial Mathematics, 5
Medial Singularities in 2D

- Generic Blum cases:
  - Derived from symmetry set
  - Index of last zero coefficient gives the order
  - Corresponding to number of sphere touches k (A^k) and order of touching j (A_j)
  - Normal point ($(A_1)^2$)
  - Branch point ($(A_1)^3$)
  - End (fold) point: g(u) has 4th order coefficient nonzero: (A_3)
  - 4 point contact not generic in 2D



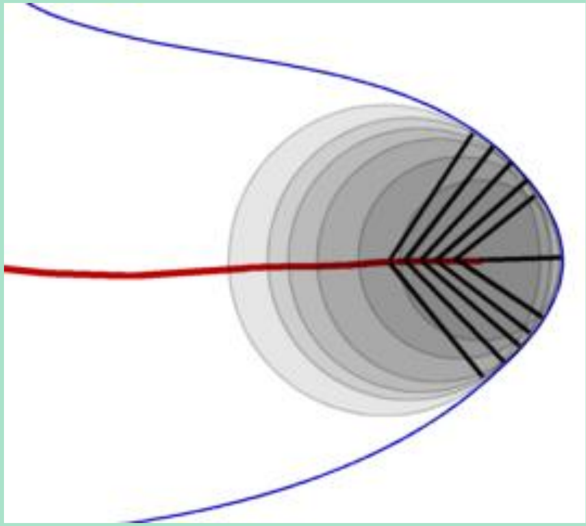
Medial Branch Point Mathematics in 2D

- Relation between angles between skeletal branch pairs and angles between 3 associated spokes
  - At the branchpoint there are three spokes
  - At all branchpoints the three surfaces meet non-smoothly
  - The medial locus bisects all three pairs of spokes
  - $\Sigma_{i=1,2,3}(\kappa_i/\psi_i)=0$; $\psi_i + \theta_i = \pi$, $i=1,2,3$; κ_i is boundary curvature

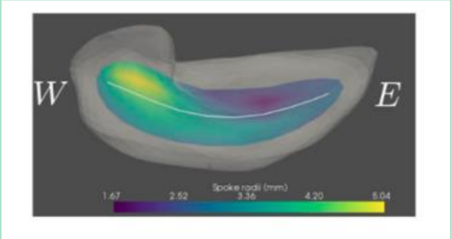


Blum Ends (Folds) in 2D

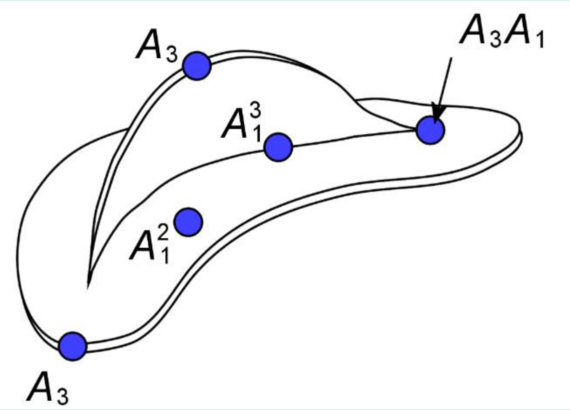
- Fold (end) atom
  - Zero object angle
  - Multiplicity 3 tangency, i.e., osculation
  - θ=0, so dr/dx = -cos(0) = -1
  - Infinitely fast spoke swing in limit

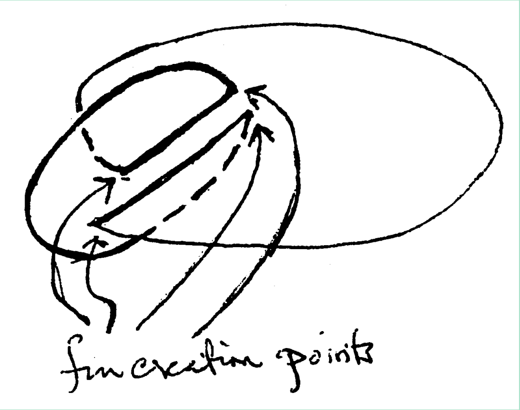


Medial Mathematics in 3D



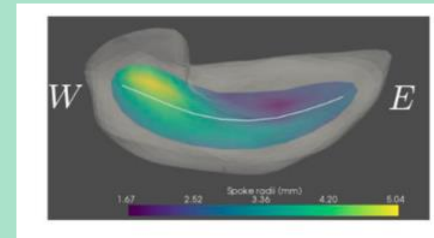
- Medial representation singularities
  - Normal point (no singularity; 2D or 3D): A_1^2 (bitangent)
  - Point on branch curve (point for 2D): A_1^3 (tritangent)
  - Point on fold curve (point for 2D): A_3 (tangent of order 3 at 1 point)
    - Surprisingly 3rd order touching at crest
  - 4 point contact not generic in 2D but is generic in 3D: A_1^4 (see next slide)
  - Ends of branch curves in 3D mix normal point and fold of branch : A_1A_3

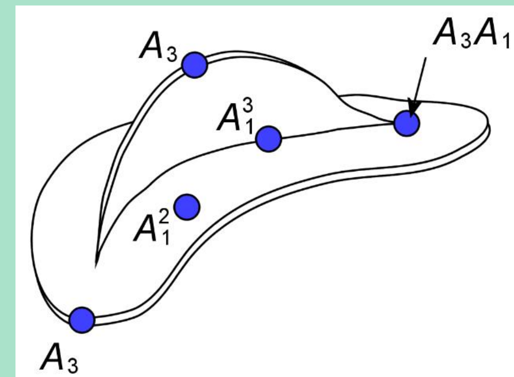


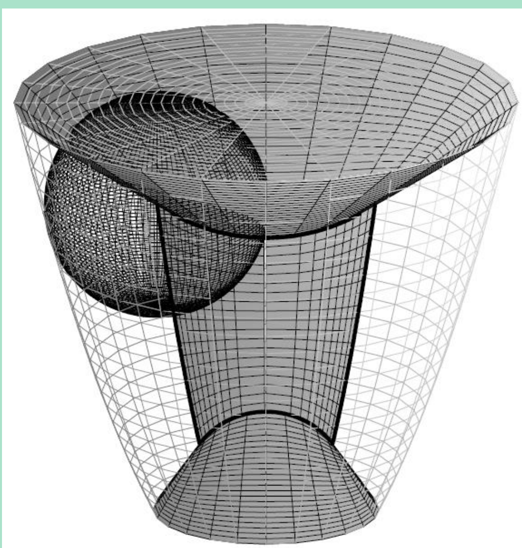


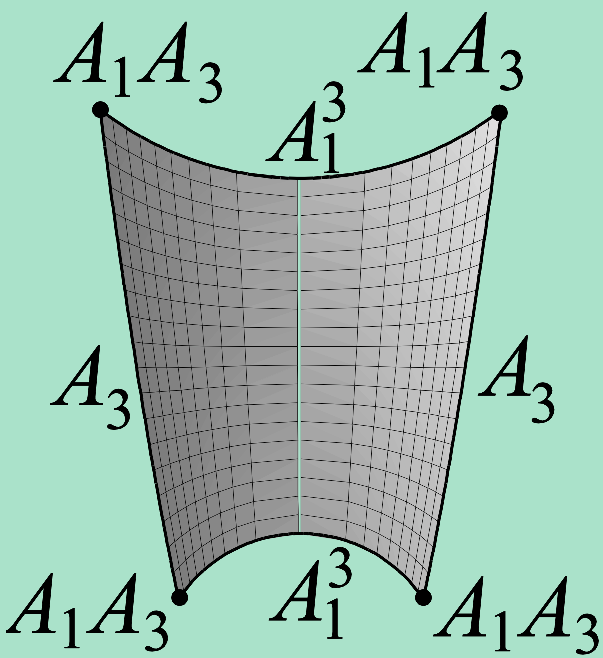
Fins in 3D

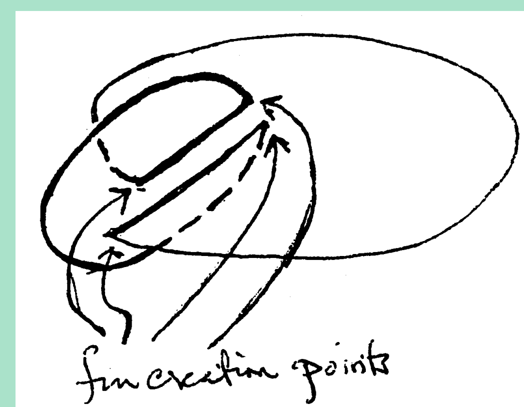
- Generic Blum cases:
  - At normal point: $(A_1)^2$
  - At branch point: $(A_1)^3$
  - At end (fold) point (crest on surface) : A_3
  - At fin end point the spoke end on the fin side is A_3 (like end) and the other is A_1: $A_3 A_1$





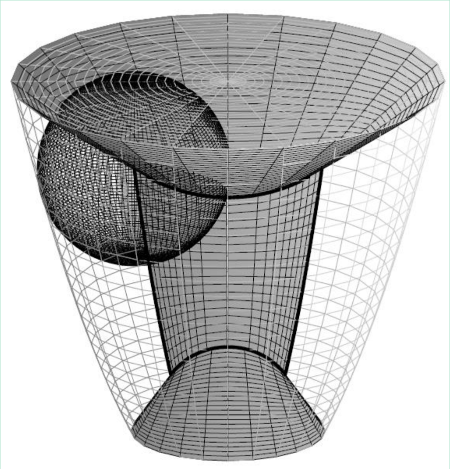




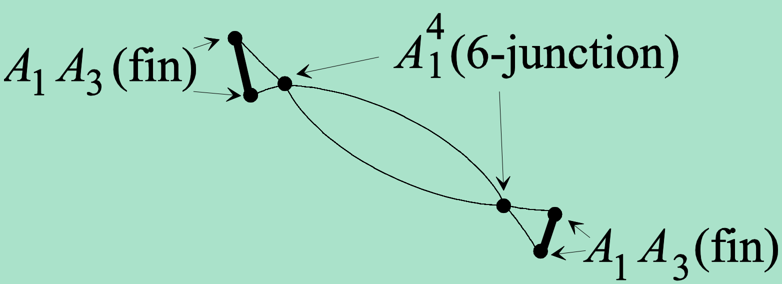


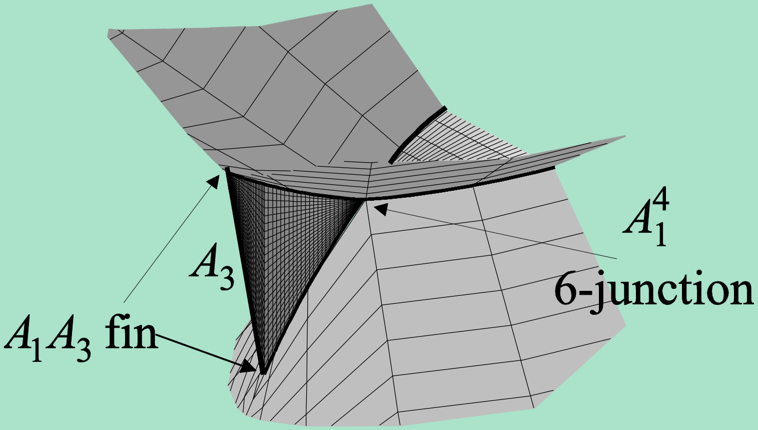
The 6-Junction: $(A_1)^4$

- Generic
- At a point only
- 4 spokes combine in pairs into 4 choose 2 = 6 branches (thus 4 crests at ends of branches)
- 4 branch curves (and their crests) intersect



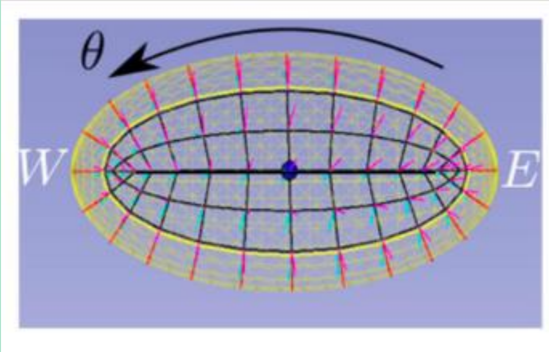
Elliptical basket, but shorten it

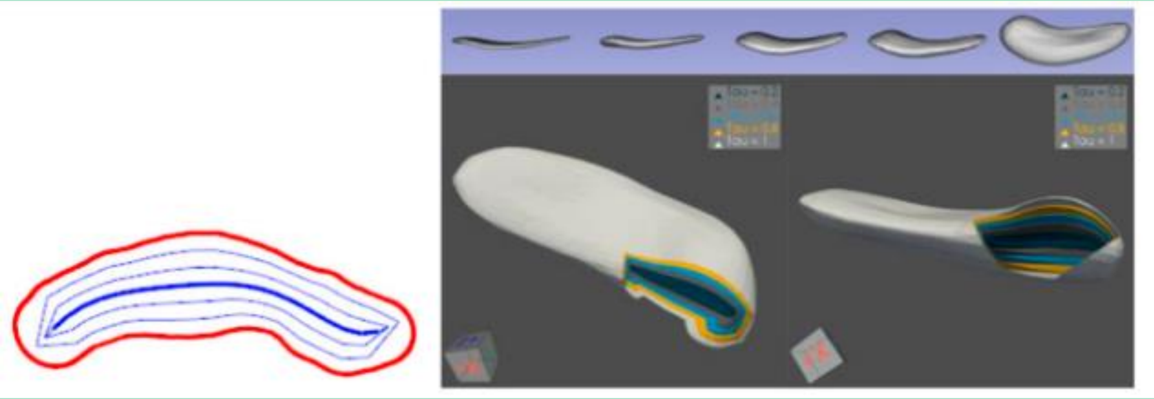




Radial Distance [Damon]

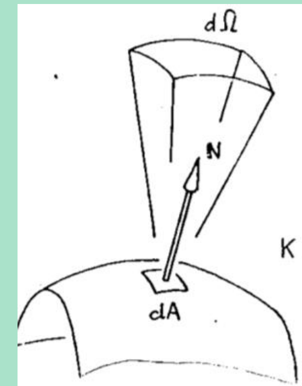
- τ_2 = radial distance = fraction of distance, along spoke, from skeleton to boundary
  - A dilation distance from skeleton, as opposed to erosion distance from boundary; different because τ_2 is spoke-length proportional
- Onion skin at $\underline{x} + \tau_2\, r\mathbf{U}$
  - Skeleton at $\tau_2 = 0$
  - Boundary (smooth) at $\tau_2 = 0$
  - For $0 < \tau_2 < 1$, onion skin has corner along fold spokes





Radial Shape Operator [Damon]

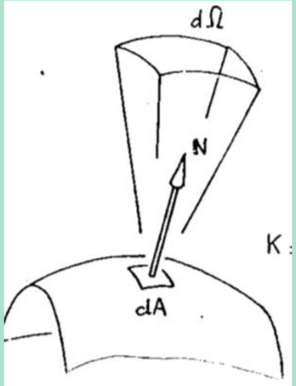
- Swing of spoke direction **U** per walking direction (u,v) on skeleton, with ($\mathbf{e}_u$, $\mathbf{e}_v$) orthogonal directions
  - Cf. original shape operator: swing of normal **N** per walking direction on boundary
    - But there Δ**N**, $\perp$ **N**, is on tangent plane
  - For spoke swing Δ**U**, $\perp$ **U**, is typically not on skeletal tangent plane
    - Project Δ**U** on skeletal tangent plane
    - Then!! express projected Δ**U** in non-orthogonal coordinates ($\mathbf{e}_u$, $\mathbf{e}_v$, **U**)



- Radial shape operator S_{rad} allows computation of projected **U** swing for any walking direction <u>w</u> on the skeletal tangent plane

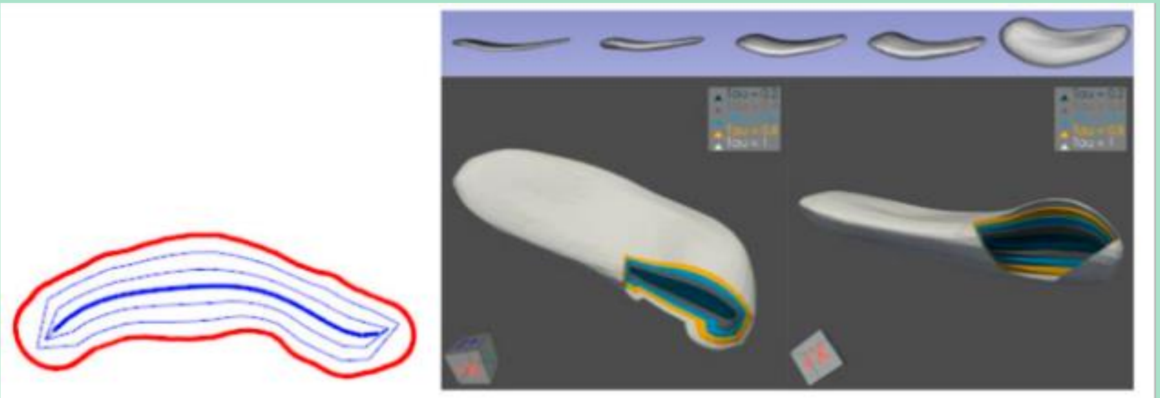
Radial Shape Operator [Damon], 2

- Swing of spoke direction **U** per walking direction (u,v) on skeleton, with ($\mathbf{e}_u$, $\mathbf{e}_v$) orthogonal directions
- Radial shape operator S_{rad} = 2×2 matrix of negative of ($\mathbf{e}_u$, $\mathbf{e}_v$) coefficients for walking directions
  $\underline{w} = w_1\mathbf{e}_u + w_1\mathbf{e}_v$ in skeletal tangent plane
- $S_{rad}\,\underline{w}$ gives projected component swing of **U**
  - Note dr(**w**) = $\mathbf{U}_{tan} \bullet \mathbf{w}$, where $\mathbf{U}_{tan}$ is component of **U** in ($\mathbf{e}_u$, $\mathbf{e}_v$)
- There is a special operator for fold points



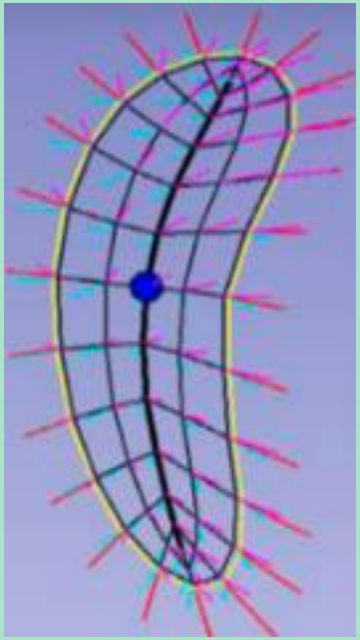
Shape Operator on Onion Skins

- Radial curvatures κ_r are eigenvalues of S_{rad}
  - $\kappa_r < 1$ to prevent spoke crossing in closed object interior
  - Are real even though S_{rad} is not symmetric
  - Boundary curvatures: $\kappa = \kappa_r / (1 - \kappa_r)$
  - Similar formula for onion skins (which have same skeleton)
- So signs of radial curvatures are the same as signs of normal curvatures at corresponding points of all onion skins
  - Thus convexity, concavity, cylindricality all correspond
  - $d\psi(\underline{x}, \tau_2) = d(\underline{x} + \tau_2 r\mathbf{U})$ carries principal radial directions to principal onion skin directions



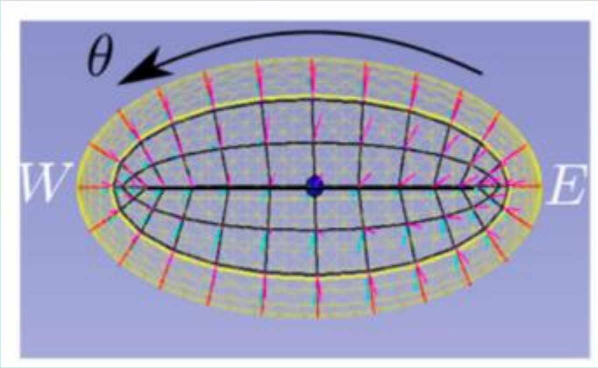
Shape Operator to Prevent Spoke Crossing

- The radial curvatures κ_{r_i} are eigenvalues of S_{rad}
- For no spoke crossing in interior (a skeletal requirement), $r < 1/\kappa_{r_i}$ for all positive radial curvatures and all skeletal points
  - A constraint on PCA-like modes
  - More likely a problem on concave boundary regions



Among Objects with with Spherical Topology, The Ellipsoid: The Primordial Shape

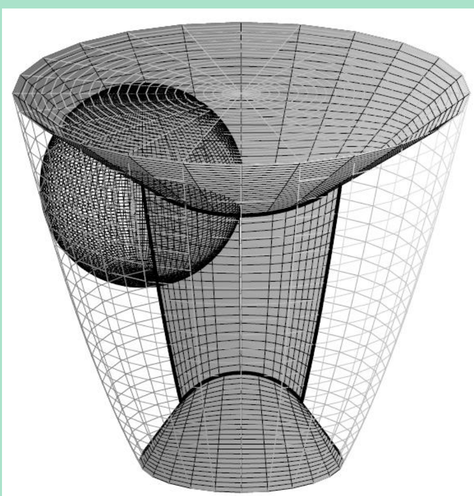
- Sphere is non-generic
  - {spheres} is only 4-dimensional: center, radius
    - Thus, it is a 0-volume subspace of {all shapes of spherical topology}
    - It has a trivial (nongeneric) skeleton: surface collapsed to a point

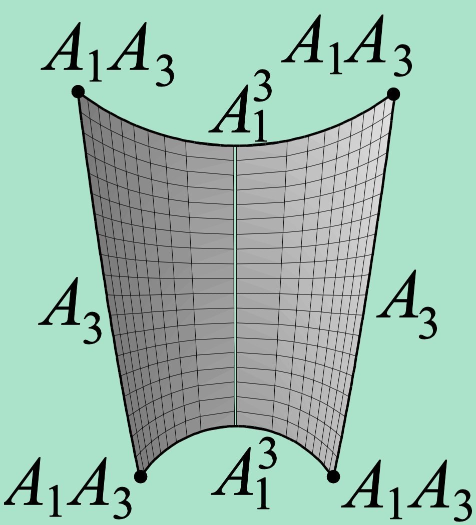


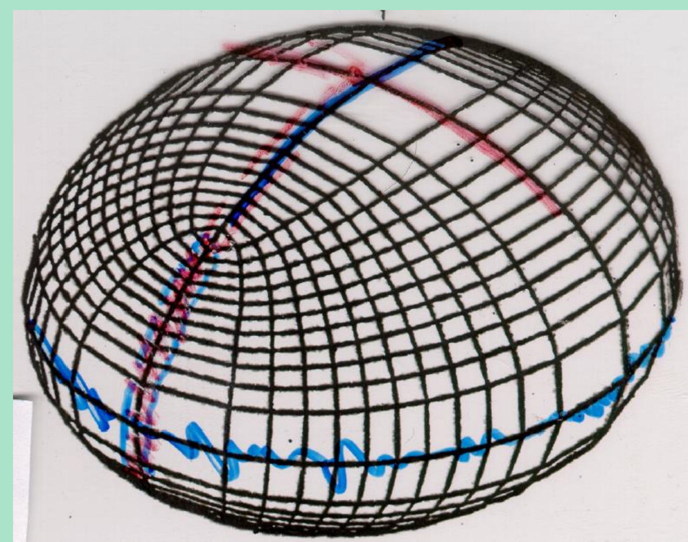
- Ellipsoid with principal radii $r_x > r_y > r_z$ is simplest shape with a skeleton in the form of a folded surface
  - Blum skeleton is ellipse in (x,y) plane with principal radii: $(r_x^2 + r_z^2)/r_x$ in x direction ; $(r_y^2 + r_z^2)/r_y$ in y direction
  - Crest (on ellipsoid) is an ellipse in the (x,y) plane
    - It has two opposing vertices

The Crest in 3D

- Boundary $\underline{b}(u,v)$ near $\underline{b}(u_0,v_0)$ and medial sphere center $\underline{x}$ are redescribed in $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{N}$ coordinates with origin at $\underline{b}(0,0)$
- At end (fold) point (crest on surface) g has 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} order coefs (of a_1 terms) zero, 4^{th} order coef nonzero: A_3
- Crest is relative max of κ_1 in $\mathbf{p}_1$ direction
- Crest curve is typically not orthogonal to $\mathbf{p}_1$ direction (the direction of the curvature maximum), i.e., not in $\mathbf{p}_2$ direction

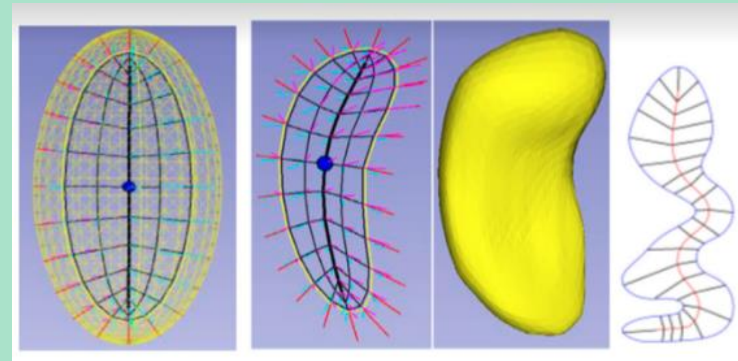


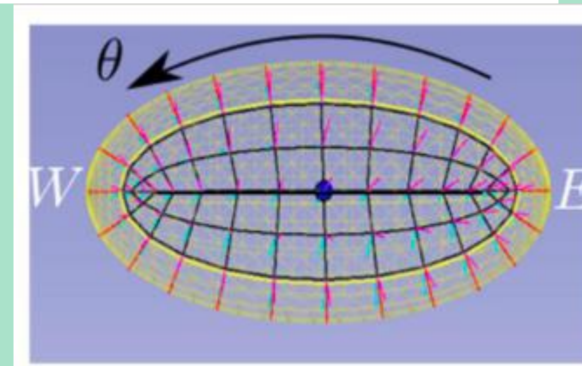




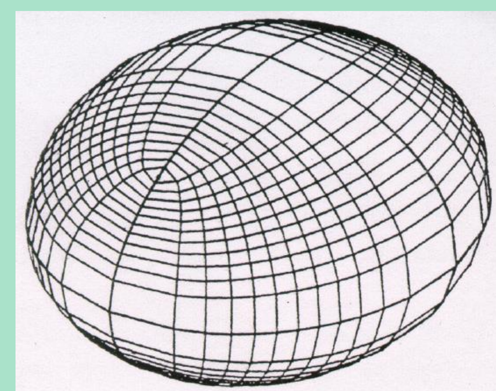
Ellipsoidal Skeletal Coordinates

- Skeleton ellipse has a 1D skeleton in 2D (the "spine"), with its spokes, radial distances τ_1
  - Spine has a zero-D skeleton, the skeletal center point
  - τ_1 needs a sign to indicate side of the spine, so $\tau_1 \in (-1, +1)$

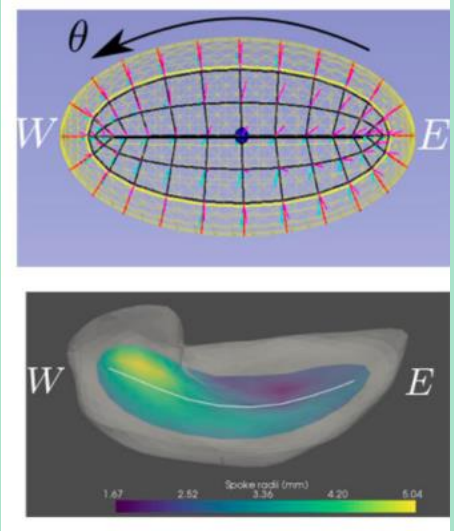




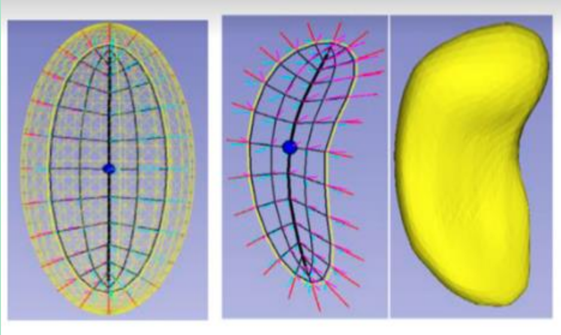
- Cyclically around skeleton from its center point ($\theta=0$) is $\theta \in (-\pi/2, \pi/2]$
- Each point in closed interior has a distinct skeletal coordinate (θ, τ_1, τ_2)

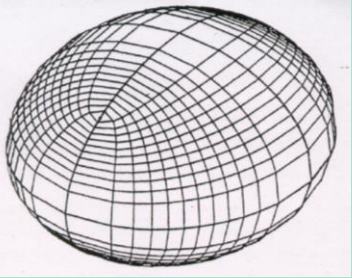


Objects with Spherical Topology and No Protrusions or Indentations



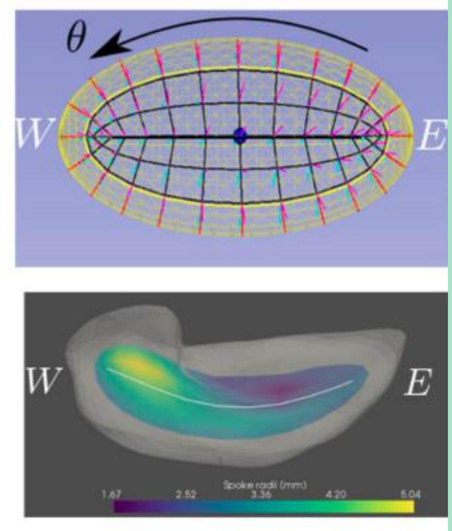
- Can be understood as diffeomorphism of the ellipsoid
  - It will have at least two opposing vertices and at least one closed crest
- Want to carry all the basic skeletal geometry into the object throughout the diffeomorphism
  - Skeletal properties to be maintained
    - Skeletal vertices must map into skeletal vertices
      - And boundary vertices into boundary vertices
    - Skeletal folds must map into skeletal folds
      - And Sboundary crests into boundary crests
  - Straight spokes must map into straight spokes
    - Ideally with τ_2 maintenance
  - Corresponding spoke lengths should stay ~equal
- **Designed to yield a strong correspondence across objects in a population via** (θ, τ_1, τ_2)

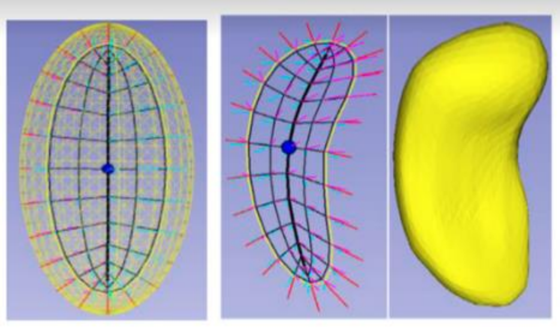


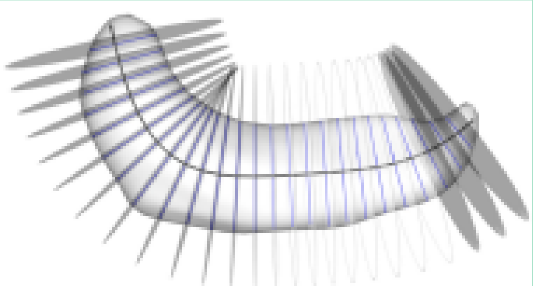


Single Figure Objects with Spherical Topology via Diffeomorphism of Ellipsoid

- Want the diffeomorphism to carry the basic skeletal geometry of the ellipsoid into the target object
  - Designed to yield a strong correspondence across objects in a population by reflecting common geometry from a common ellipsoid
- Because the skeleton is designed to carry the curvature of the interior of the object, it appears not possible for the spokes across the skeleton (with their radial distance τ_1), which are straight in the ellipsoid's skeleton to remain straight in target object skeleton.
  - But note Taheri's swept plane skeleton, in later slides

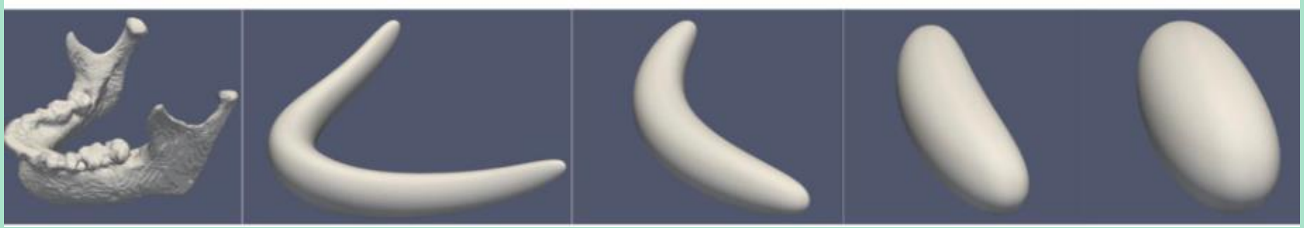


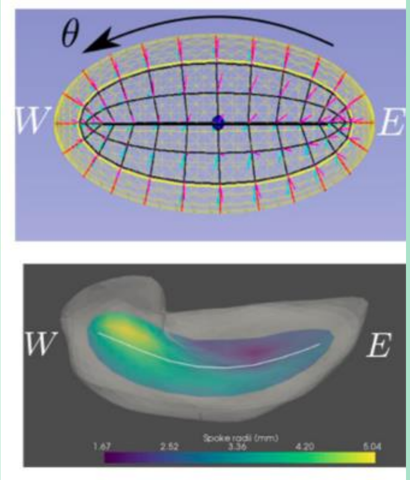




Single Figure Objects with Spherical Topology via Diffeomorphism of Ellipsoid, 2

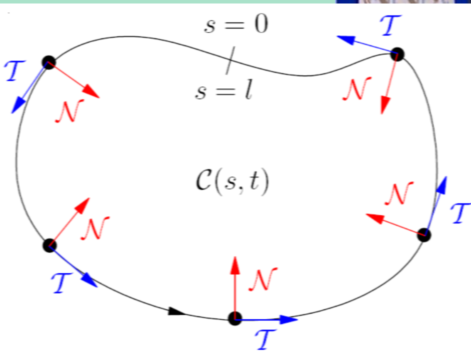
- Computation of diffeomorphism from ellipsoid to object can be initialized with curvature-smoothing flow of target object boundary, which will approach ellipsoid





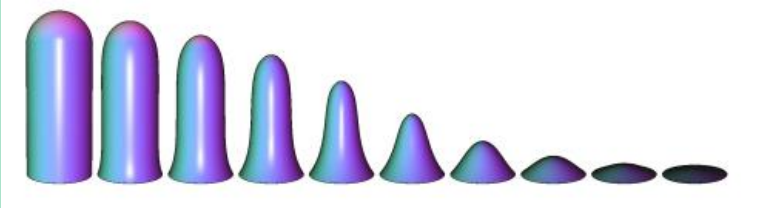
  - Before it approaches its limiting sphere
  - By conformalized mean curvature flow
    - But produces poor correspondence for skeletal geometry maintenance
  - Will collapse protrusions and indentations early (see subfigure discussion)
  - Its inverse, the desired diffeomorphism, needs to be modified to maintain the basic ellipsoidal skeletal geometry for the object (see next slide)

Conformalized Mean Curvature Flow of an Object Boundary



- Original idea was mean curvature flow:
  - $d\underline{b}/dt = H(\underline{b}, t)\,\mathbf{N}(\underline{b}, t)$
    - t is time of flow
  - Though it does deform boundary into a near ellipsoid, it collapses regions of high curvature into a point,
    - i.e., has singularities

- Improved method does not have singularities: conformalized mean curvature flow [Kazhdan]



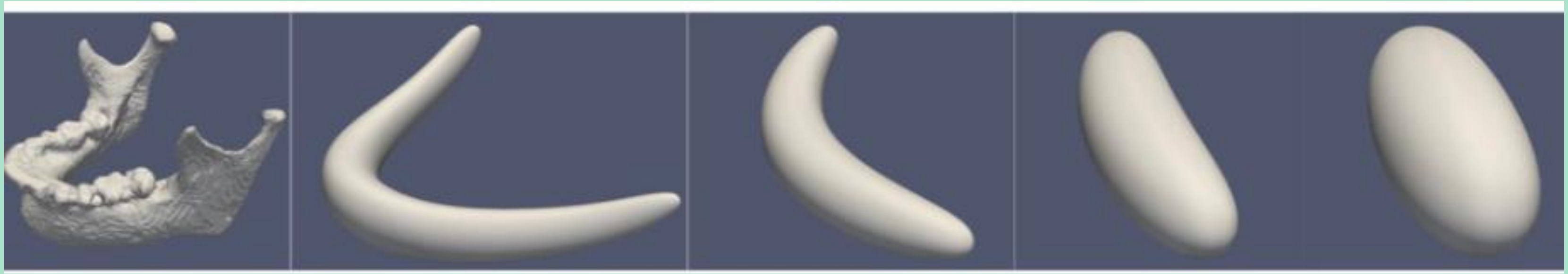
  - Changes the metric for the flow:

$$\tilde{g}_t = \sqrt{\left|g_0^{-1} g_t\right|}\, g_0.$$

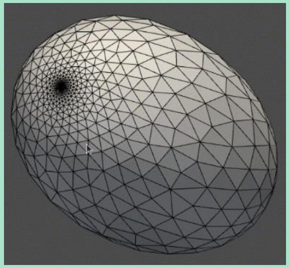
    - Metric is in principal coordinates
    - Metric changes with deformation time t

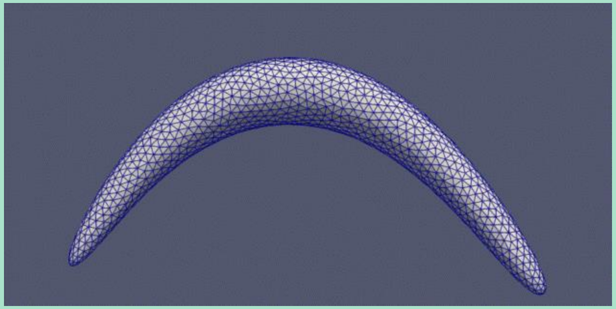
Conformalized Mean Curvature Flow of an Object Boundary, 2

- Method does not have singularities: conformalized mean curvature flow [Kazhdan]



- But it still takes boundary mesh with equal-area elements into an ellipsoid mesh with greatly different-area elements
  - Concentration of mesh elements near vertices and from subfigure regions, dependent on protrusion & indentation noise
  - Does not map vertices into vertices, nor crests into crests

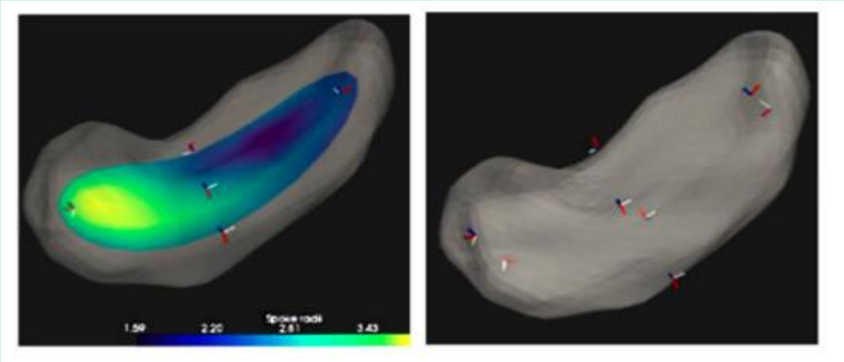


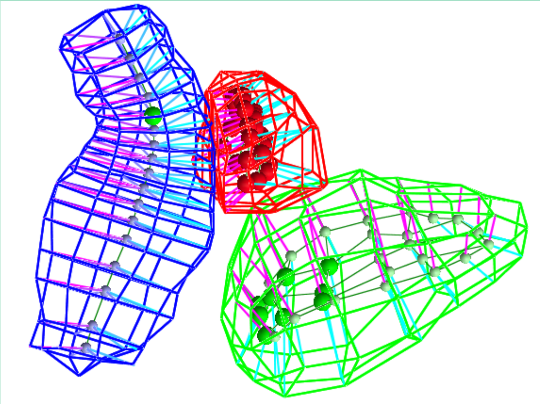


- That is, it produces poor correspondence

Fitted Frames for Single-Figure Objects

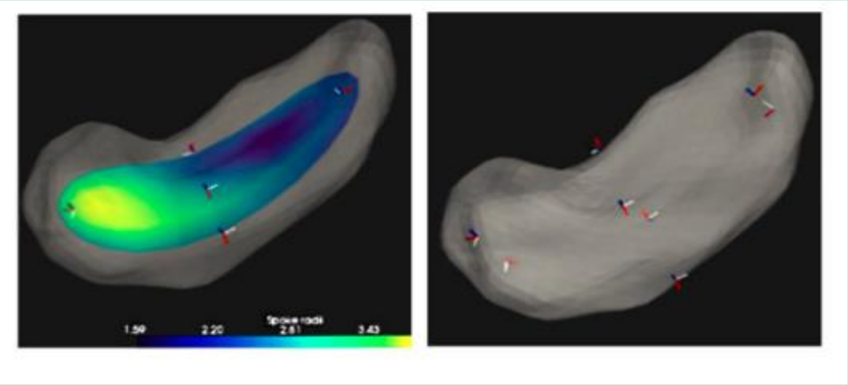
- Objective
  - Like fitted frames for boundary (Cartan), carry local geometry,
    - **But here of the interior, not just the boundary**
    - Like boundary fitted frames, capture local curvatures,
  - But also provide a coordinate system for distant geometry
    - E.g., vertex direction vs. object center direction
    - E.g., relation of centers of related objects
  - Do it with good correspondence across the object population



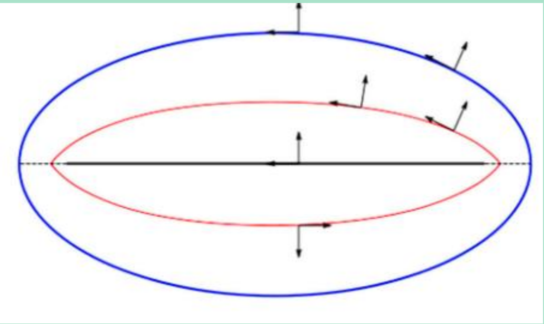


Fitted Frames for Ellipsoid, 2

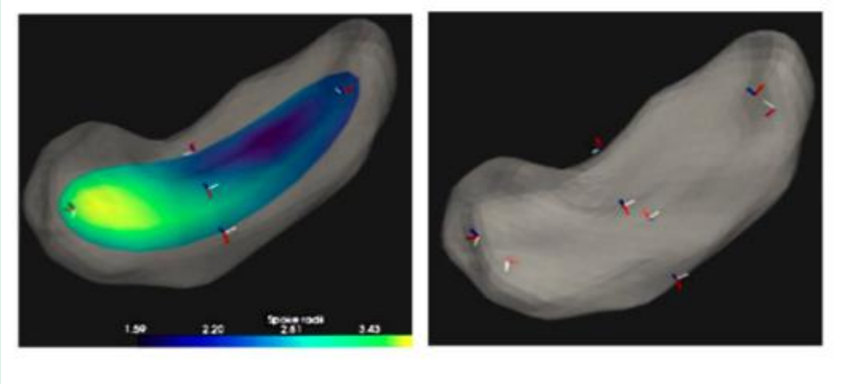
- Important additional capability
  - Important for statistics, allows features that are alignment independent
    - Frame rotations: local relative compass
    - Location differences in local frame: local relative ruler
    - Later: inter-object relations
- Avoiding difficulties of alignment, because of its dependence on scale



Fitted Frames for Ellipsoid, 3

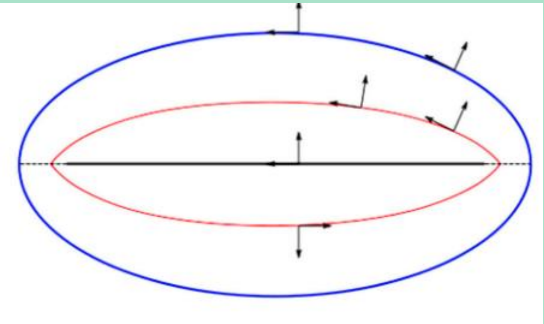


- On onion skins
  - Thus on skeleton (τ_2=0)
    - Respecting side of fold, dependent on θ
    - And thus on spine (τ_1= τ_2=0)
  - Thus on boundary (τ_2=1)
  - In 2D normal and tangent to onion skin form frame
  - In 3D
    - Third frame vector $\mathbf{f}^3$ is normal to onion skin
    - Second frame vector $\mathbf{f}^2$ is along fixed τ_1 as θ varies
    - $\mathbf{f}^1 = \mathbf{f}^2 \times \mathbf{f}^3$

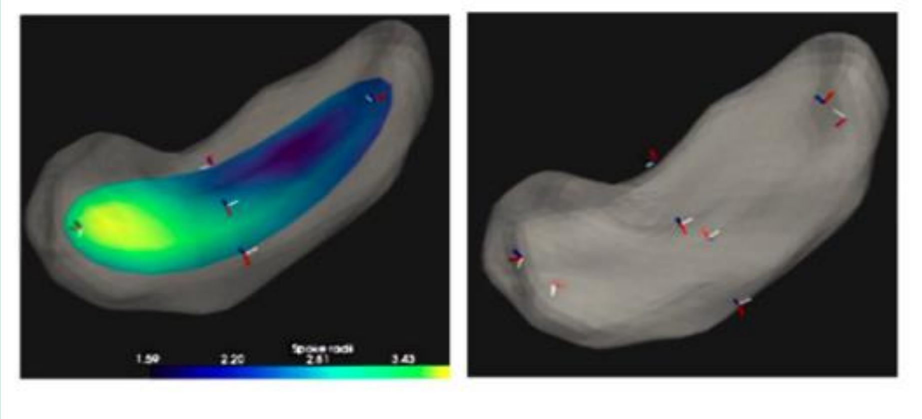


- Allows spoke interpolation
  - Rotations of frame
  - r interpolation recognizing dr properties

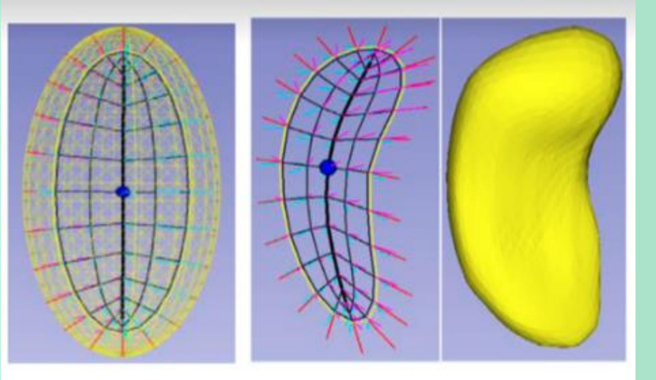
Fitted Frames for 3D, Single Figure Object of Spherical Topology



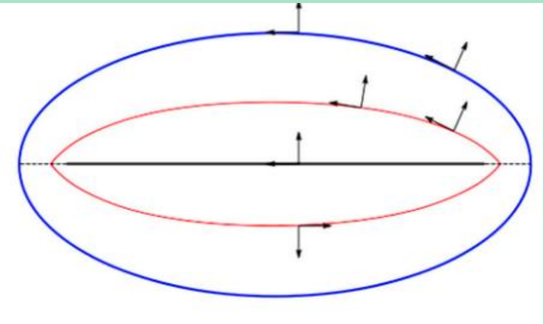
- With proper diffeomorphism, same definition as from ellipsoid
  - In 3D
    - Third frame vector $\mathbf{f}^3$ is normal to onion skin
    - Second frame vector $\mathbf{f}^2$ is along fixed τ_1 as θ varies
    - $\mathbf{f}^1 = \mathbf{f}^2 \times \mathbf{f}^3$



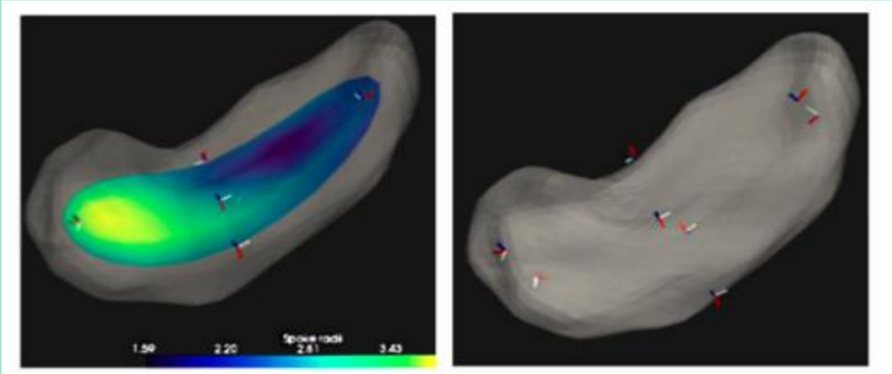
- Approximation by carrying $\mathbf{f}^1$ and $\mathbf{f}^2$ by diffeomorphism from ellipsoid

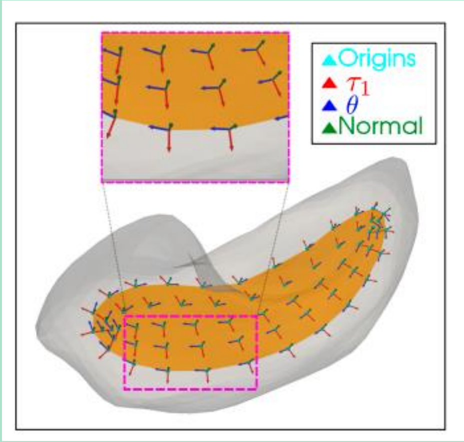


Affine Fitted Frames for 3D Single Figure Object of Spherical Topology



- Carry $\mathbf{f}^1$, $\mathbf{f}^2$, and $\mathbf{f}^3$ by diffeomorphism from ellipsoid
  - Will no longer have unit lengths
    - Lengths form features
  - Will no longer be mutually orthogonal
    - Angles form features

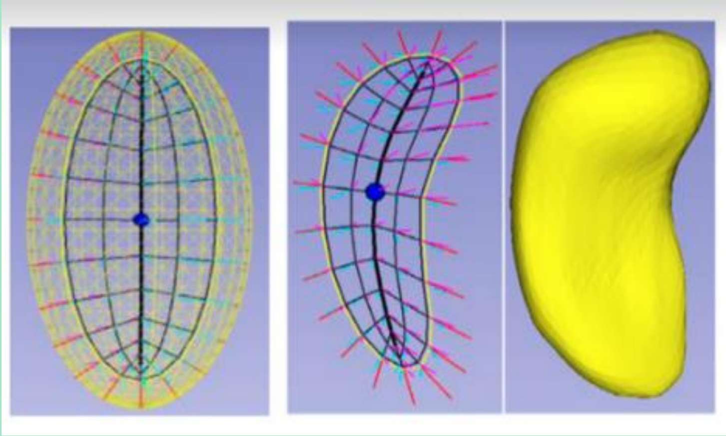


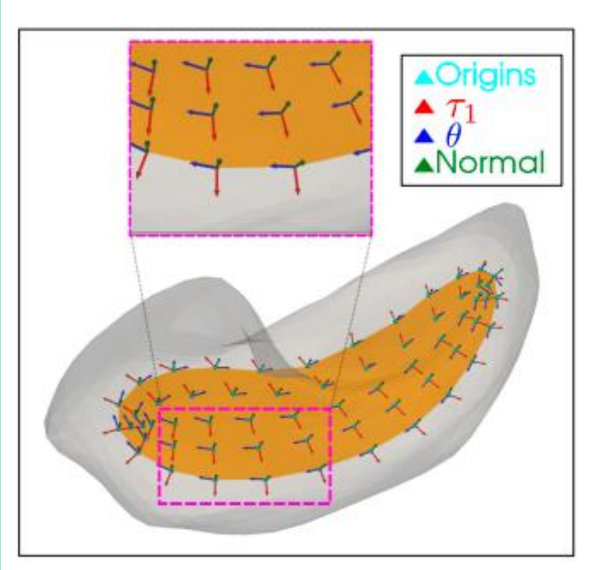


Affine fitted frames to a hippocampus skeleton [Z Liu]

Skeletal Features for Single Figure Object

- Skeletal positions
  - Ideally relative to
    - Center point frame, or
    - Neighbor skeletal position frame
- Spoke lengths
- Affine frame lengths
- Directions
  - Frame vector directions
  - Affine frame directions
  - Ideally, all relative to local frame
- For statistics, spoke directions and frame rotations are Euclideanized using PNS

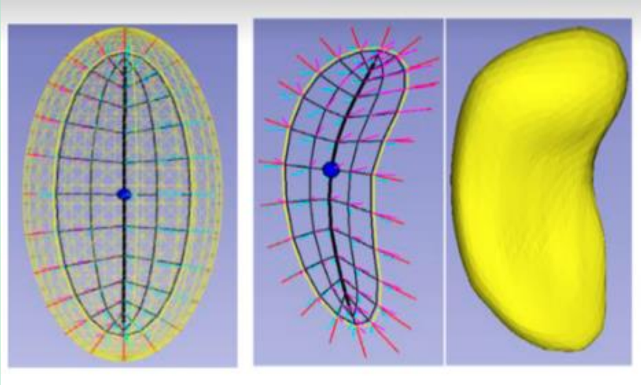




Affine fitted frames to a hippocampus skeleton [Z Liu]

Summary of Production of Skeletal Features with Correspondence

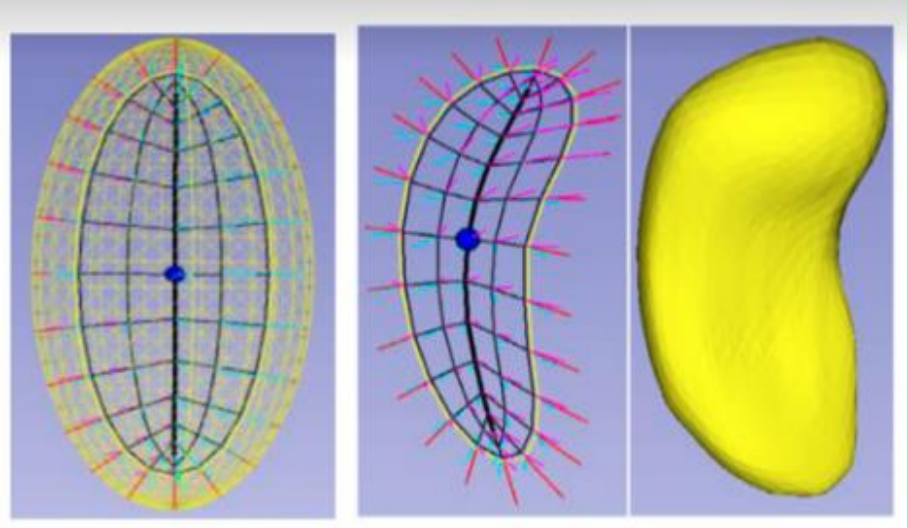
- Let rep'n of each training sample come from diffeomorphism of the same ellipsoid, recognizing
  - Vertices and crests
  - Boundaries, using CMC flow
  - Spoke loci and radial lengths
    - From skeleton to boundary
    - From spine to skeletal fold



- And producing correspondence via skeletal coordinates (θ, τ_1, τ_2)
  - Achieved by fitted frames via onions skins
    - With directions and positions measured via local frames
- Avoids alignment by use of frames fitted to onion skins

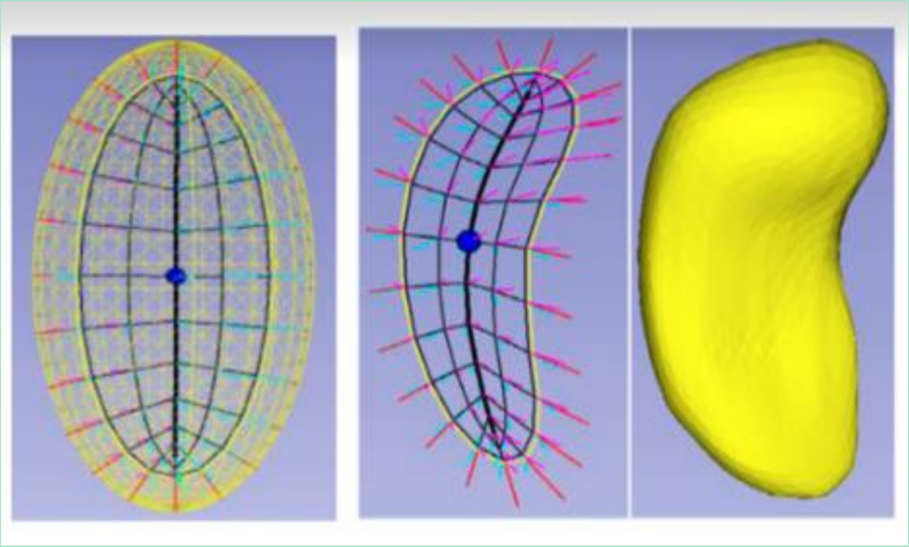
S-reps

- **Hold on**: Diffeomorphism of a skeleton will not be medial
  - That's great because it avoids bushiness
  - But now *generalized skeleton*,
    not necessarily precisely in the middle between point pairs
  - And with spokes to the boundary that are not necessarily orthogonal ("partial Blum")
    but still do not cross and fill the interior
  - Generated by optimization of "fit" to the boundary
- !!Damon: most of the medial geometry relations still work
  - Radial distance
  - S_{rad}
  - Radial curvatures
  - Spoke interpolation
  - A few of the geometric relations require "partial Blum"



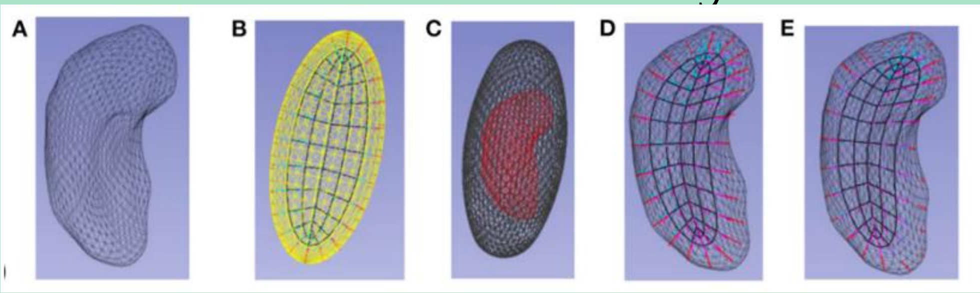
Discrete S-reps

- Sampled in (θ, τ_1), typically uniformly in each
  - Each has a location $\underline{x}$, but normally understood relative to neighboring location in terms of local fitted frame
  - Each has a spoke length r
  - Each has a spoke direction **U**, normally understood relative to the fitted frame at $\underline{x}$



Fitting an S-rep to a Boundary Mesh

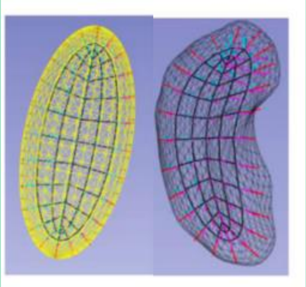
- Fitting rather than generated from boundary to fix branching topology
- Fitting to boundaries



  - Optimization [Z Liu]
    - Stage 1: approximate diffeomorphism to yield correspondence
    - Stage 2: Refinement optimization: Penalties:
      - 1) foremost, a term heavily penalizing crossing of the spokes, via κ_{r_i}
        - » Could be a hard constraint
      - 2) the deviation of the implied boundary from the target object boundary;
      - 3) the deviation of the angle of the spokes from the corresponding boundary normal
      - Could use the difference in corresponding spoke lengths
    - Code at slicersalt.org
  - Alternatives on next slides

Fitting an S-rep to a Boundary Mesh, 2

- Fitting to boundaries via proper diffeomorphism
  - From ellipsoid fit to approximate ellipsoid produced by flow
    - This an optimization
  - By temporal stages producing reverse diffeomorphsisms to yield stage to stage small deformations
    - Each of these is an optimization (see next slide)
    - So it too produces approximate correspondence



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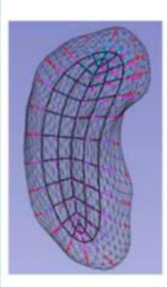
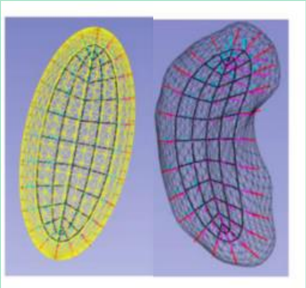


- So this fitting approach also produces approximate correspondence

Fitting an S-rep to a Boundary Mesh, 3

- By temporal stages producing reverse diffeomorphsisms to yield stage to stage small deformations



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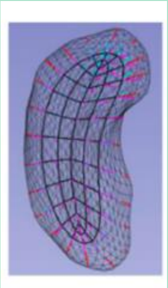
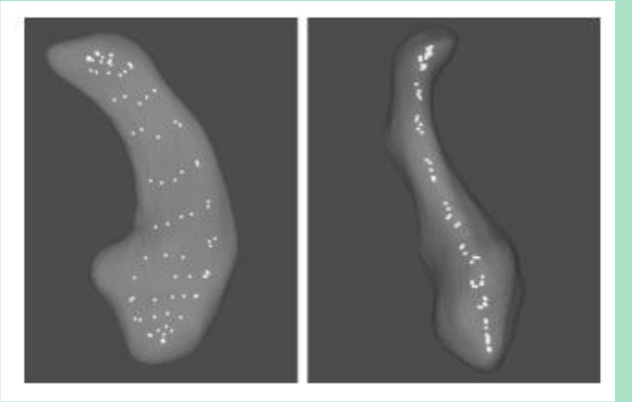


  – Each of these is an optimization: Penalties:
    - 1) foremost, a term heavily penalizing crossing of the spokes, via κ_{r_i}
      – Could be a hard constraint
    - 2) the deviation of the implied boundaries between stages
    - 3) the deviation of the angle of the spokes from the corresponding boundary normal
    - 4) the deviation of the implied crest to the later stage crest
      – Also for vertices
    - 5) the difference in corresponding spoke lengths
  – In late stages of development [Tapp-Hughes]

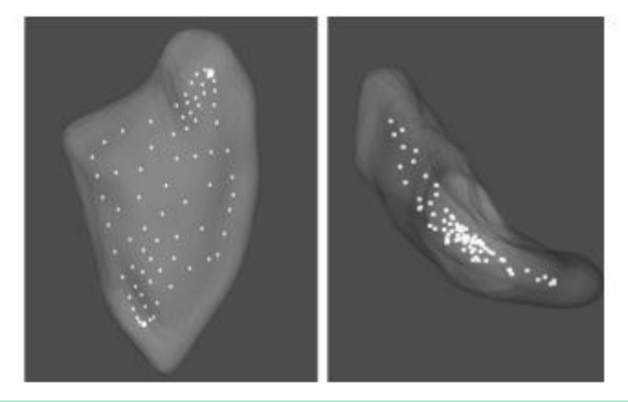
Fitting an S-rep to a Boundary Mesh, 4

- By Multilayer Perceptron [Ninad Khargonkar]
  - Input is point cloud P from boundary mesh
  - Based on PointNet++
  - Trained from deformed ellipsoids



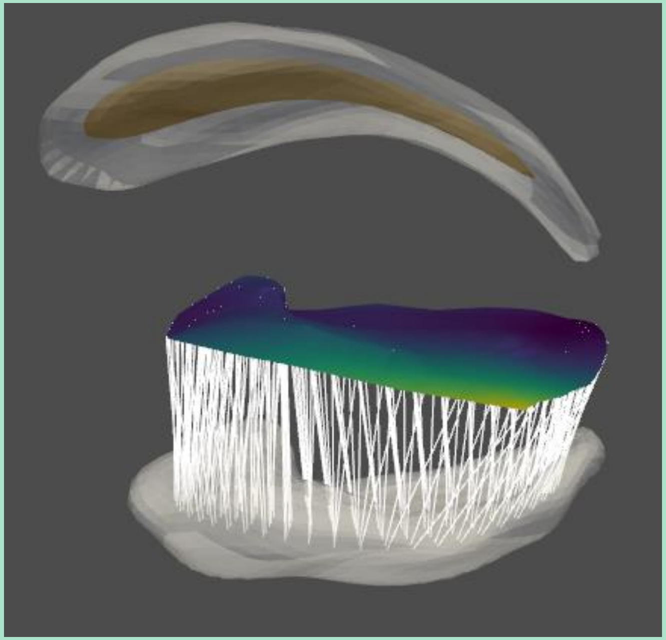
- Representation: Weights matrix W
  - $S=W^TP$
  - $r(s_j) = \Sigma_{ij} W_{ij} d_i$; $d_i = min_{s_j \in S} ||p_i - s_j||_2$
- Loss functions:
  - Dist $S_{grd=1}$ to S_2: $\frac{1}{|S_1|}\sum_{a\in S_1}\min_{b\in S_2}||a-b||_2^2 + \frac{1}{|S_2|}\sum_{b\in S_2}\min_{a\in S_1}||b-a||_2^2$
    - S_{grd} is initial model
  - Mediality $\Sigma_{s,\text{ closest 3 } p} |p-s| - r(s)$
  - Spread regularization: $-avg_{ij} | s_i - s_j |$
- Spoke construction:

$$spoke(s_j) = \frac{(p_{i_j^*} - s_j)}{||p_{i_j^*} - s_j||_2} \text{ where } i_j^* = \arg\min_{i=1,\dots,N} W_{ij}$$



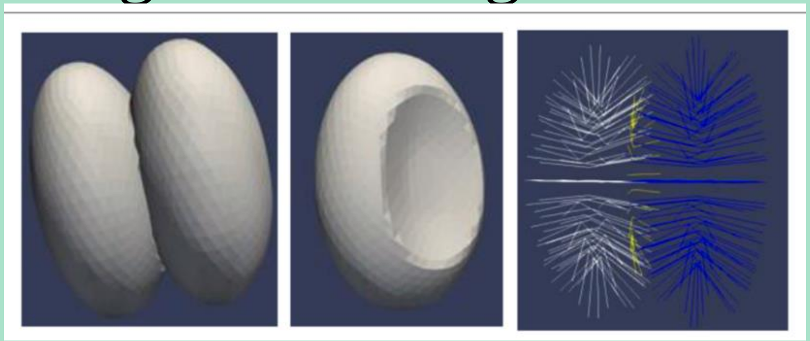
Multi-object S-reps

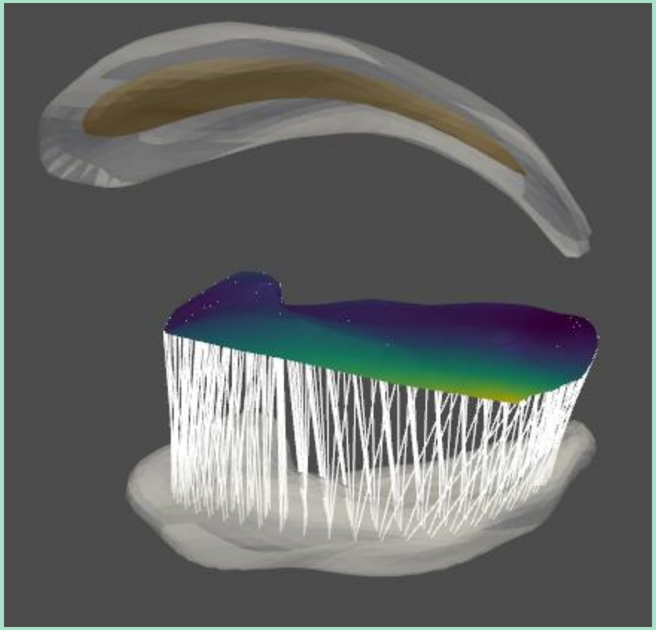
- Need to represent each object and relation between objects
  - Inter-object linking surface; **corres-pondences across a population depend on the objects' correspondences** [Z Liu]
  - Roughly bisecting linked objects, but as a single surface folded only once
  - Non-crossing links from each object to the linking surface
    - Accomplished by spline fits to selected non-crossing links
  - So far, developed only for 2 more or less parallel objects: hippocampus and caudate
    - But produced superior classification than concatenation of the 2 objects

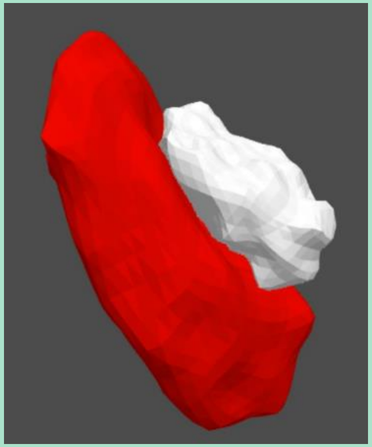


Multi-object S-reps, 2

- See multi-entity statistics, esp. correlated features
- Abutting objects [Krishna]
  - Consistent fitted frames at abutting surface regions
  - Inter-object linking surface shares those surface regions
    - With zero link length
  - So far, developed only for two more or less parallel objects:
    - Hippocampus & caudate
    - putamen & globus pallidus



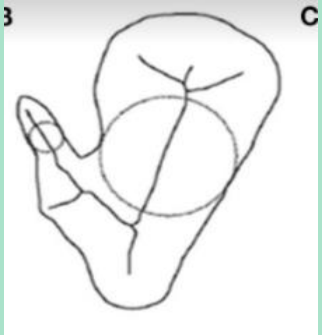




Cm-reps

- Paul Yushkevich lectures
  - PDE method
    - Read his paper with name “PDE: on 790-6 on my google drive
  - Deformation
    - Read his paper with name “deformation” on 790-6 on my google drive
    - Note ability to handle multifigure (medial branching) in 3D

Multi-figure S-reps



- Connecting the Blum skeleton pieces into a host – subfigure tree [Katz]
  - Human vision analog and thus multiscale
    - Only developed for 2D
    - Based on saliency measure
    - Uses skeletal width as a ruler
    - Integrates "visual potential", incl. from nearby attached subfigures

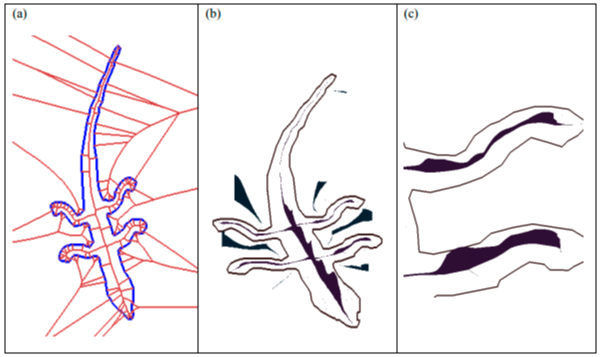
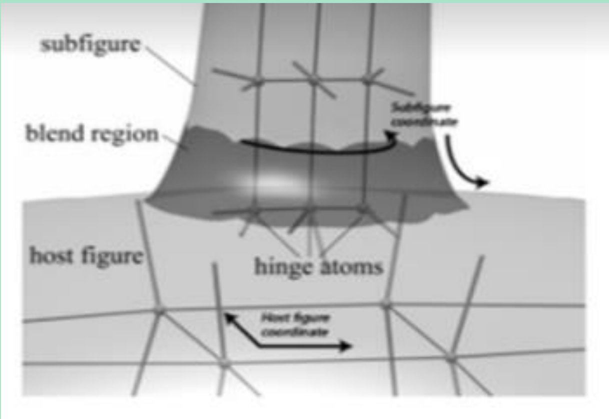


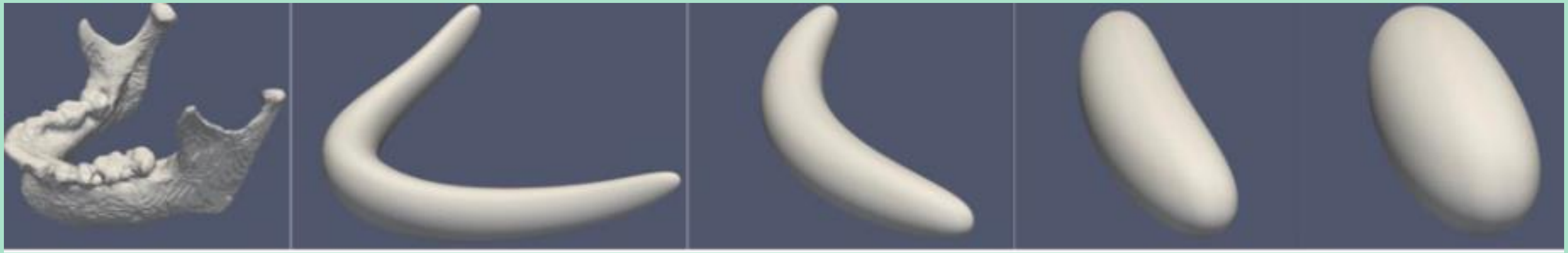
Fig. 4.13 Saliency Results of a Lizard Object

(a) A lizard object, and (b) it's local visual significance. (c) shows a magnified view of the legs demonstrating the increased saliency at the ends of the figures.

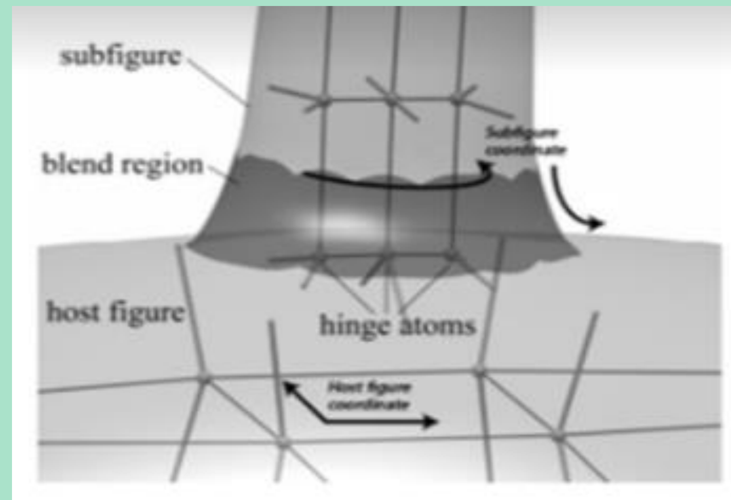
- In 3D
  - Explicit s-rep connections [Q Han]
    - See next slide
  - Use medial branching [Yushkevich]

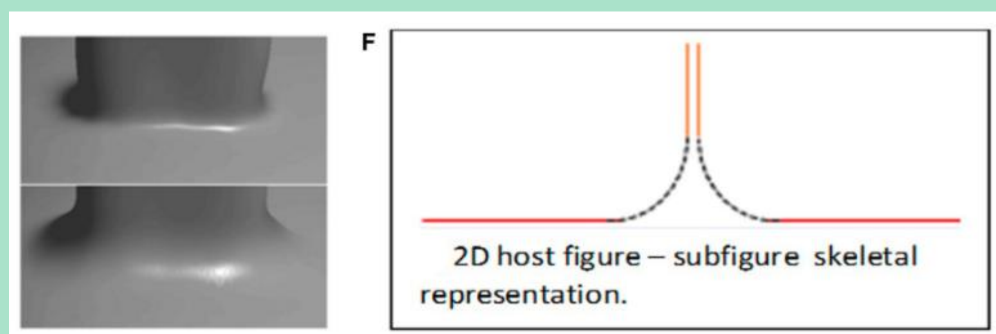


Multi-figure S-reps, 2



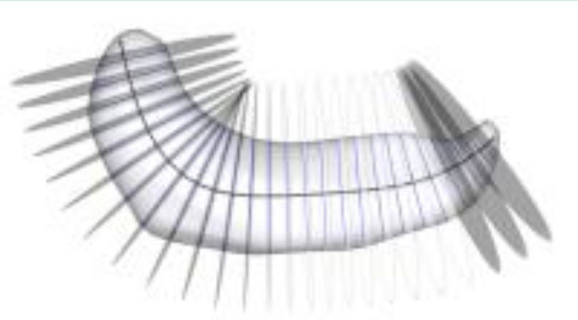
- Detection via disappearance during smoothing flow [TBA]
- Explicit s-rep connections [Q Han]
  – Protrusion or indentation has its own s-rep, truncated
  – Special neck and skeleton form connecting subfigure and host figure
  – Subfigure designed by following smoothing from target object





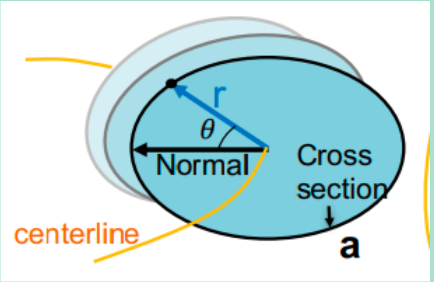
Swept-Plane S-reps [Taheri]

- Based on Damon definition of a slab
  - Has continuous series of cut planes that do not intersect in the interior of the object



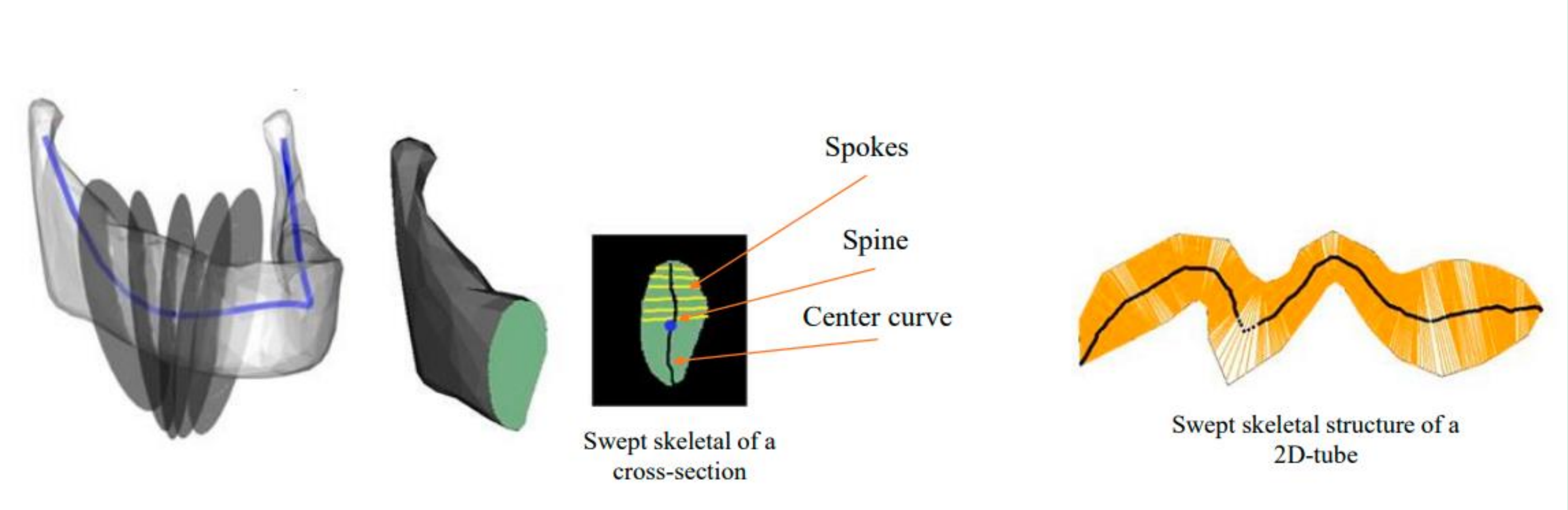
- 3D skeleton combines the 2D skeletons on the cut planes
  - Spine formed from locus of planar center points
  - Thus views object as a generalized cylinder
    - Spine creation needs to satisfy Damon's relative curvature condition for a generalized cylinder:

$$r(\theta, s) < \frac{1}{k(s)cos(\theta)} \quad \text{when } cos(\theta) > 0.$$



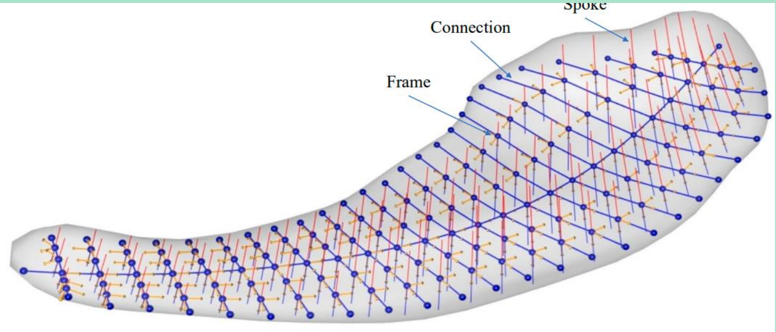
Swept-Plane S-reps [Taheri], 2

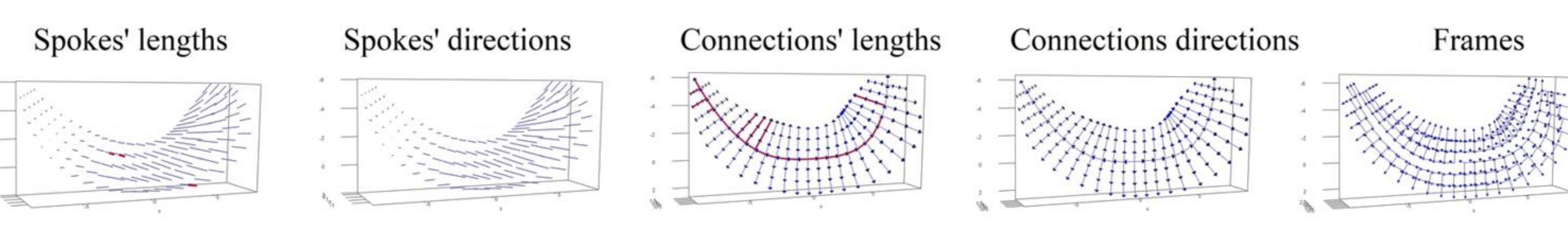
- Fitting method is in development
- The cut planes and their spine
  - For each spine point
    - Collinear on-skeletal spokes from spine to fold
    - Each spoke from skeleton also in that plane
    - So maintenance of both spoke types and radial lengths can be maintained by diffeomorphism from ellipsoid



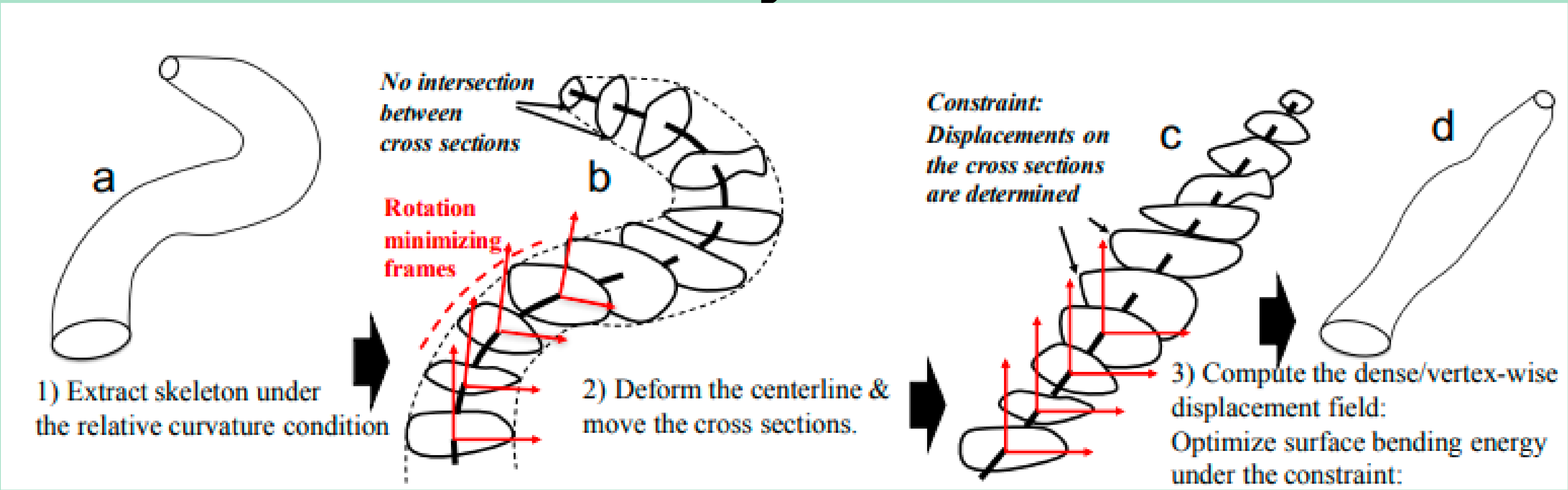
Swept-Plane S-reps [Taheri], 3

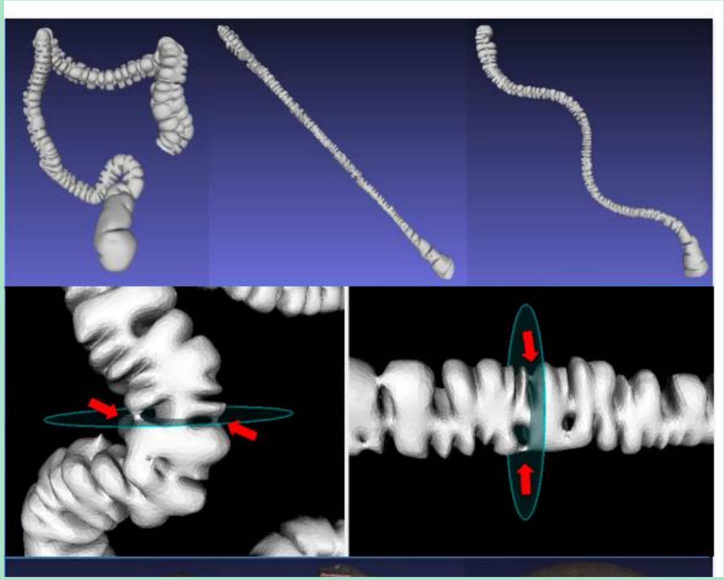
- Fitted frames
  - One element constant over slice
- Thus allows statistics, e.g., hypothesis tests by locality and feature type
  - Hypothesis test results on hippocampi between 6-month olds exhibiting later autism or not (red means significant difference)

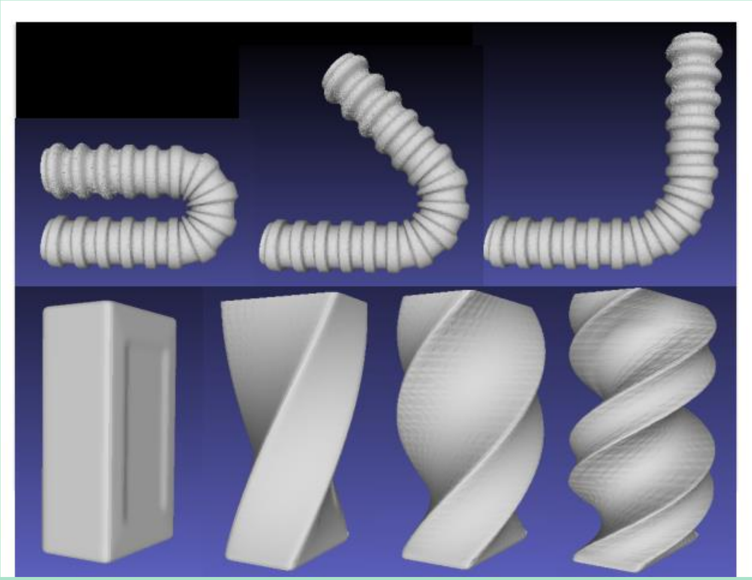




Straightening a Generalized Cylinder [Ma]



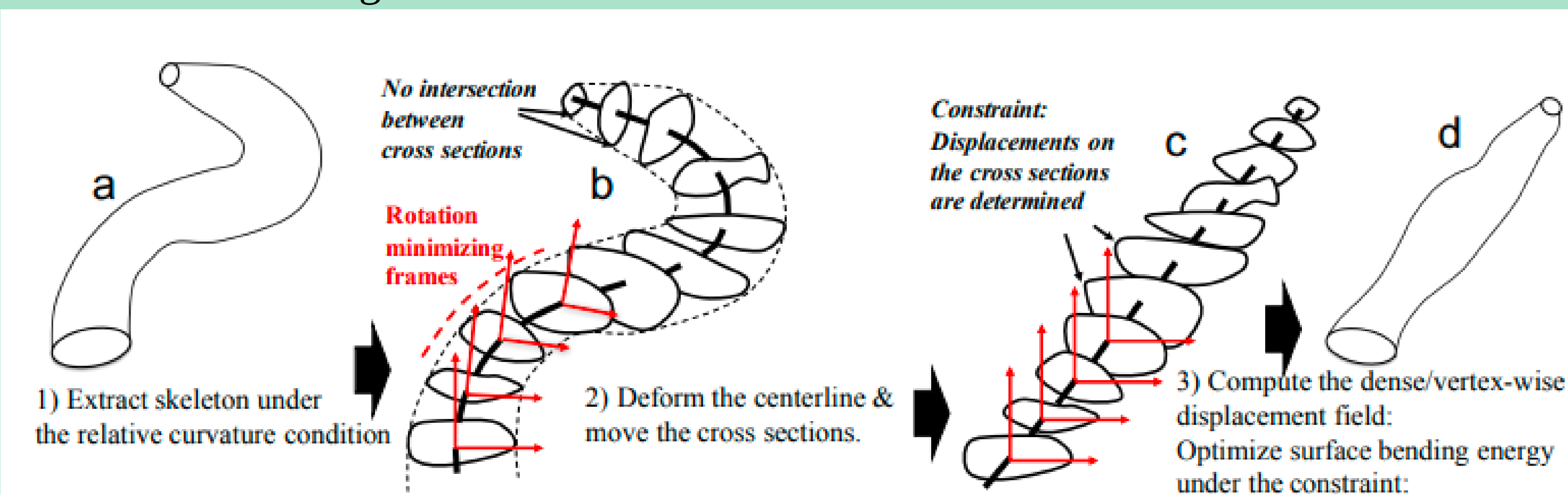




Straightening a Generalized Cylinder [Ma], 2

- Details of step 3
  - On sparsely sampled cross sections
    - Minimization of frame rotation to target axis shape
  - Using constraint on sampled cross sections
    - Minimize local thin shell bending energy on mesh points
      - Uses Laplace-Beltrami operator, which characterizes change of mean curvatures

$$E_{total} = \sum_{j\in\{x,y,z\}} \left(\sum_{i=1}^{n} (\Delta f^{j}(v_i^{unknown}))^2 + \sum_{i=1}^{m} (\Delta f^{j}(v_i^{fixed}))^2\right)$$



Finite Element Models Based on Skeletal Modeling [Crouch]

- Both host figure and protrusion subfigures
- Multiscale subdivision by spoke interpolation and radial distance subsampling
  - Thus force application iterative: large scale to small