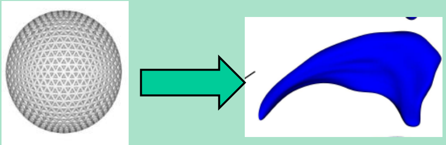
Shape Representation Geometry and Topology

- Surface normal directions and tangent directions
- Curvatures: curves and surfaces
- Number of holes & connected components topology
- Shape spaces
- Manifolds and geodesics
- Distance measures
  - Riemannian metrics

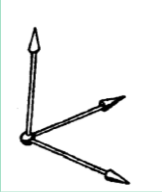
- Figures taken from Koenderink, *Solid Shape*

Mathematics: Local Surface Geometry

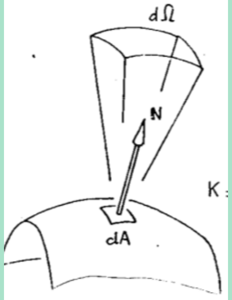
- Refs: O'Neill: *Elementary Differential Geometry*, Koenderink: *Solid Shape*



- Surface parameterization via (u,v): $\underline{x}(u,v)$
- Frame: In n-space, n orthogonal unit vectors ordered w/ right-handed rule



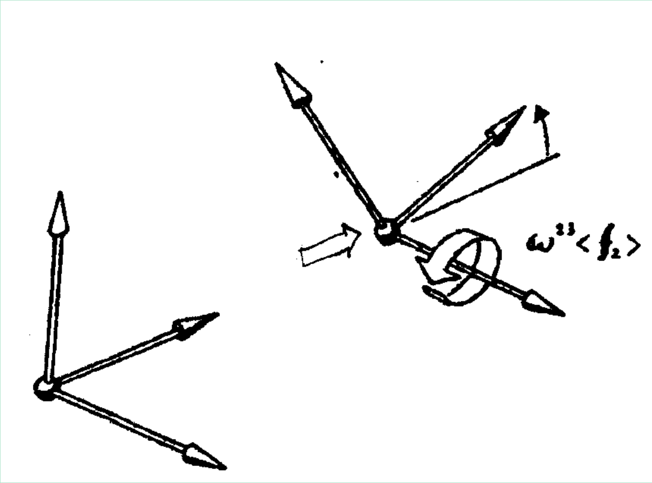
- Tangent vectors: $D_u\underline{x}(u,v)$, $D_v\underline{x}(u,v)$
- Normals: $\mathbf{N}(u,v) = D_u\underline{x}(u,v) \times D_v\underline{x}(u,v)$ normalized to unit length



- Tangent plane is spanned by $D_u\underline{x}(u,v)$, $D_v\underline{x}(u,v)$, so it has normal $\mathbf{N}(u,v)$
- Gaussian fitted frame: two orthogonal unit vectors $\mathbf{f}^1$ and $\mathbf{f}^2$ in tangent plane together with normal $\mathbf{f}^3$

Mathematics: Local Normals Geometry

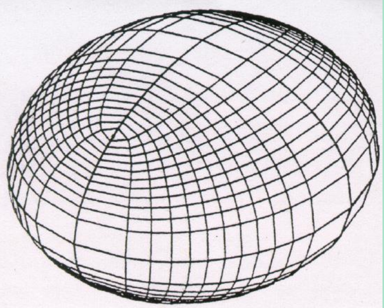
- Directional derivatives (swing) of normals
  - For walking direction $\mathbf{w}$, $D_{\mathbf{w}}\mathbf{N}(u,v)$
  - Fitted frames to surface; $\mathbf{f}_3 = \mathbf{N}$, $\mathbf{f}_1$ and $\mathbf{f}_2$ in tangent plane
  - With walking direction $\mathbf{w} = t_1\, \mathbf{f}_1 + t_2\, \mathbf{f}_2$ in tangent plane, $D_{\mathbf{w}}\mathbf{N}(t_1 \text{ in } \mathbf{f}_1, t_2 \text{ in } \mathbf{f}_2) = k\mathbf{w} + \tau\mathbf{w}^{\perp}$
    - k = “normal curature” or “nosedive”;
    - τ = “geodesic torsion” or “twist”
  - When walking in direction $\mathbf{w}$, $\mathbf{N}$ swings into $k\mathbf{w} + \tau\mathbf{w}^{\perp}$, i.e., about hinge $\mathbf{c} = (k\mathbf{w} + \tau\mathbf{w}^{\perp})^{\perp}$



Mathematics: Local Surface Geometry

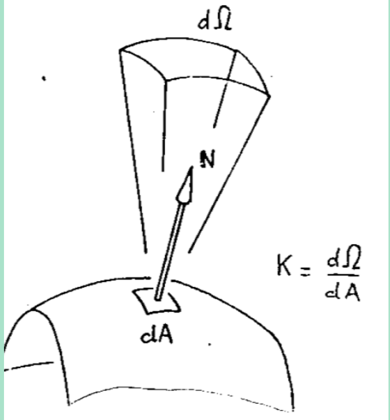
- With $\mathbf{w} = t_1\,\mathbf{f}_1 + t_2\,\mathbf{f}_2$, $D_{\mathbf{w}}\mathbf{N}(t_1 \text{ in } \mathbf{f}_1, t_2 \text{ in } \mathbf{f}_2) = k\mathbf{w} + \tau\mathbf{w}^{\perp}$
  - If $\mathbf{f}_1=\mathbf{w}$, $D_{\mathbf{w}}\mathbf{N}(u,v) = k_1\mathbf{w} + \tau_1\mathbf{w}^{\perp}$
  - Then $\mathbf{f}_2=\mathbf{w}^{\perp}$, $D_{\mathbf{w}^{\perp}}\mathbf{N}(u,v) = k_2\mathbf{w} + \tau_2\mathbf{w}^{\perp}$, but $\tau_1 = \tau_2$
  - $D_{[\mathbf{f}1\ \mathbf{f}2]^T}\mathbf{N}(u,v) = M_{II}(u,v)[\mathbf{f}1\ \mathbf{f}2]^T$, with

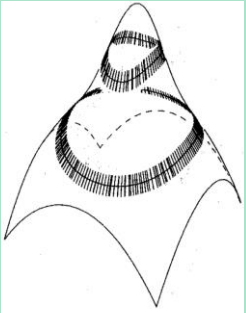
    $$M_{II} = \begin{bmatrix} k_1 & \tau \\ \tau & k_2 \end{bmatrix}$$, a symmetric matrix
  - M_{II} is called the “shape operator”
- At every surface point $\exists$ a tangent frame $\mathbf{p}_1$, $\mathbf{p}_2$
  - With in each frame direction there is no twist, i.e., pure nosedive:
    - Rotate $(\mathbf{f}_1, \mathbf{f}_2)$ to diagonalize M_{II}
    - $D_{\mathbf{p}1}\mathbf{N}(t_1\ t_2) = \kappa_1\,\mathbf{p}_1$ and $D_{\mathbf{p}2}\mathbf{N}(t_1\ t_2) = \kappa_2\,\mathbf{p}_2$
  - $\mathbf{p}_1$ and $\mathbf{p}_2$ are called “principal directions”
  - κ_1 and κ_2 are called “principal curvatures”
  - $M_{II}(u,v)$ and $\mathbf{p}_1(u,v)$ determine all curvatures at all (u,v)



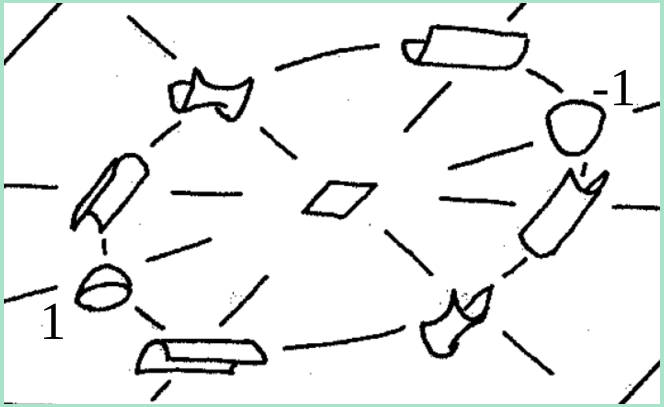
Surface Curvatures and Regions

- Important summaries of curvature at a surface point
  - Gaussian curvature $K = \kappa_1 \kappa_2 = \det(M_{II})$
    - Unit sphere areal swing of normal per unit area on surface
    - Sign of K:
      - >0: convex or concave
      - =0: cylindric ("parabolic")
      - <0: saddle-shaped ("hyperbolic")



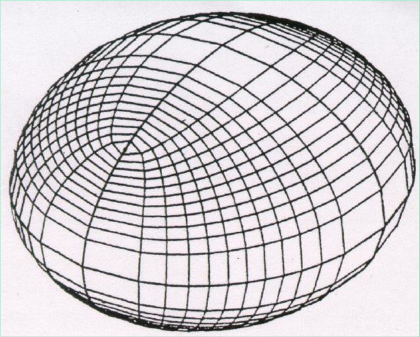


  - Mean curvature $H = (\kappa_1 + \kappa_2)/2 = \mathrm{tr}(M_{II})/2$
    - Mean of the k values over all walking directions
    - Sign of H distinguishes convex from concave when K>0
  - Curvedness $C = \ln[(\kappa_1^2 + \kappa_2^2)^{½}]$;
  - Shape type $S \in [-1,1]$: from concave through concave cylindrical through hyperbolic through convex cylindrical through convex on (κ_1, κ_2) graph



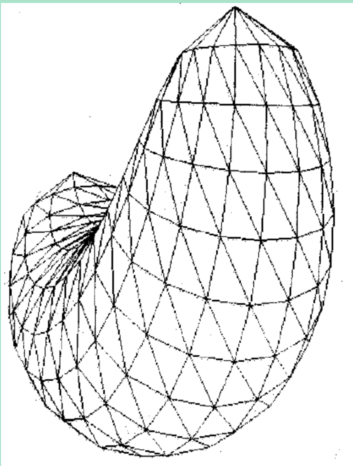
Ridges: Crests and Troughs

- Crest point
  - Relative min of negative κ (most sharply curving in a 1D convex fashion) along principal curve of associated **p**



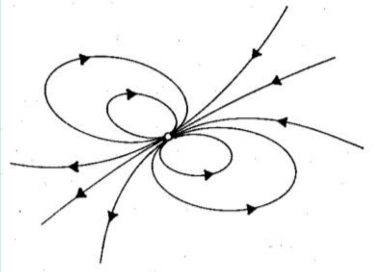
- Trough point
  - Relative max of positive κ (most sharply curving in a 1D concave fashion) along principal curve of associated **p**

- From crest to trough, principal curve passes through a parabolic curve (κ =0)



Mathematics: Local Surface Topology

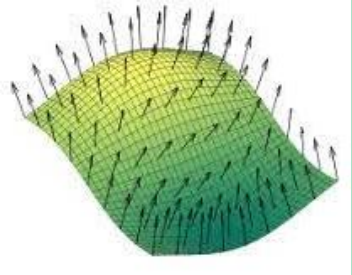
- Number of holes in a closed surface
  - Genus g of the surface; does not change under diffeomorphisms
  - Euler number $\chi = 2 - 2g$
- Sum, over points where a smooth vector field on a surface is zero, of winding number (how many counterclockwise swings the vector field swings as you pass counterclockwise around the point $= \chi$

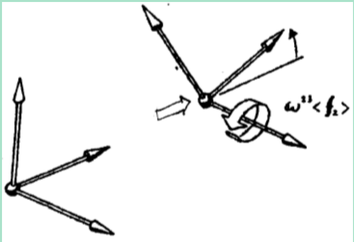


  - So principal curves on objects with spherical topology (no holes; χ =2) must have singular pts: locally spherical ($\kappa_1 = \kappa_2$)
- Connected components in set S: Maximal subsets of S э: between every pair of points in a connected component ∃ a path between the points that stays within the component = a maximal subset of topological space S that cannot be covered by the union of two disjoint open sets

Shape Spaces

- An object representation understood as an abstract manifold (smooth surface; see next slide)
  - Examples
    - Any plane or hyperplane (locus of points with Euclidean metric)
    - {n points on a 2D surface centered at zero and with average distance squared =1} = S^{3n-4}
    - {normal at one point on a 2D surface} = S^2
    - {n normals on a 2D surface} = $(S^2)^n$
    - {1 fitted frame on a 2D surface} = hemisphere of S^3 = set of rotations SO3

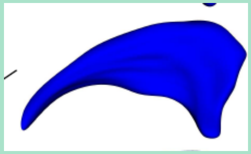


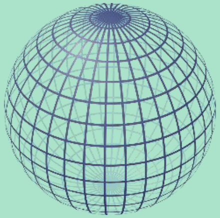


    - Set of diffeomorphisms of an object (ignoring finiteness crit'n)
- Statistics are taken over manifolds
  - Examples: means, covariances, principal directions

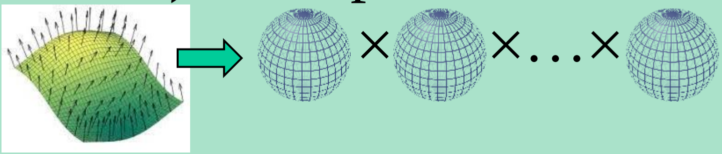
Manifolds

- Manifold: at an open set about any point there is a best fitting tangent plane on which any derivative of deviation from the tangent plane can be taken
  - Examples
    - An object in 3-space if its surface is smooth





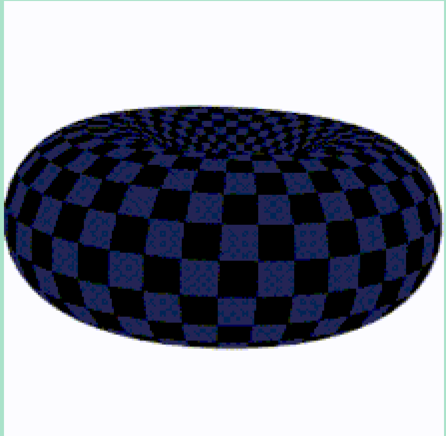
    - Unit 2- sphere: all possible directions in 3-space
    - Unit 3-sphere: all possible frames (rotations) in 3-space
    - Polysphere: Cartesian product of spheres, i.e., a sequence of directions

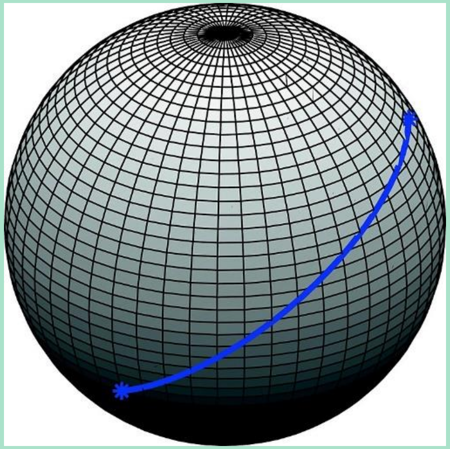


  - Mapping from points p on the manifold to the tangent plane is called Log_p
  - Inverse is called Exp_p
  - Log_p for orthogonal projection carries surface arc lengths as metric on the tangent plane

Geodesics on a Manifold

- Geodesic
  - Locally shortest path in given starting direction
    - Minimization, so computation involves solving a differential equation
  - Depends on the metric
    - E.g., Euclidean metric on a plane
  - On a manifold, for every starting point with a starting direction, there is a fixed geodesic path of increasing length
  - For every pair of points on a manifold, there are 1 or more geodesics connecting the points; each has a length; the one with shortest length is the shortest connecting path





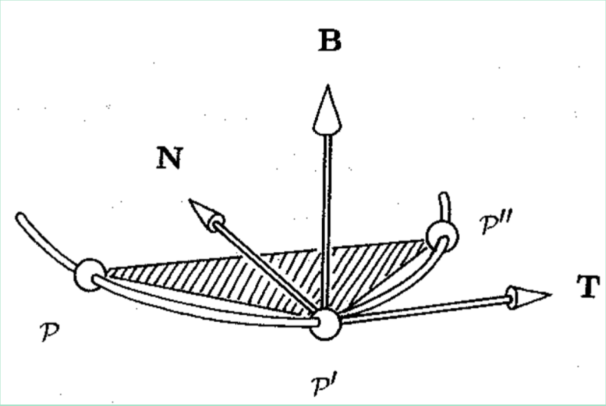
Riemannian Manifold

- Distances squared in an infinitesimal path in some direction at some point is a sum of weighted squared distances over elements of some frame
  - $D^2 = \Sigma j\ w_j\ d_j^2$, with j running over frame directions

- In a Euclidean (hyperplane) all weights are 1
- Metric of mapping $Exp_{\underline{p}}$ from tangent plane at $\underline{p}$ to a smooth manifold is Riemannian
- Metric tensor specifies the distance operator
  - $n \times n$ symmetric matrix with non-negative eigenvalues
  - Eigenvectors $\mathbf{v}_i$ form the specified frame and matrix V
  - Eigenvalues w_i form the weighting factors determining distance
  - Metric tensor: $G = V \Lambda V^T$; $\langle \mathbf{a},\mathbf{b} \rangle = \underline{a}^T G \underline{b}$; $\underline{||a||^2 = a}^T G \underline{a}$
  - M_{II} forms metric tensor when mapping from a surface point to its tangent plane

Mathematics: Local Curve Geometry

- $\underline{x}(s)$ with s being arclength
- Fitted (Frenet) frame
  - $\mathbf{f}_1$ = tangent $\mathbf{T} = d\underline{x}(s)/ds$; has length 1
  - $\mathbf{f}_2$ = normal $\mathbf{N}$ = normalized $d\mathbf{f}_1(s)/ds$ |
    = normalized $d^2\underline{x}(s)/ds^2$; has length 1
    - Positive curvature κ such that $d\mathbf{f}_1(s)/ds = \kappa\, \mathbf{N}$
    - $\mathbf{f}_1$ and $\mathbf{f}_2$ span best-fitting plane, in which circle with radius $1/\kappa$ is best-fitting circle to curve



  - $\mathbf{f}_3$ = binormal $\mathbf{B} = \mathbf{f}_1 \times \mathbf{f}_2$
    - Torsion τ (signed out of plane curvature) $= -d\mathbf{B}/ds \bullet \mathbf{N}$
      $= -\, d\mathbf{f}_3(s)/ds \bullet \mathbf{f}_2$