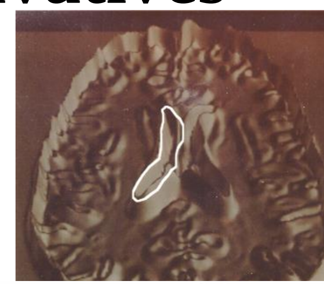
The Math Needed to Understand Image Processing cont.

- Two aspects of scale
  - Levels of detail
  - Gaussian apertures and spatial scale
  - Intensity noise vs. scale

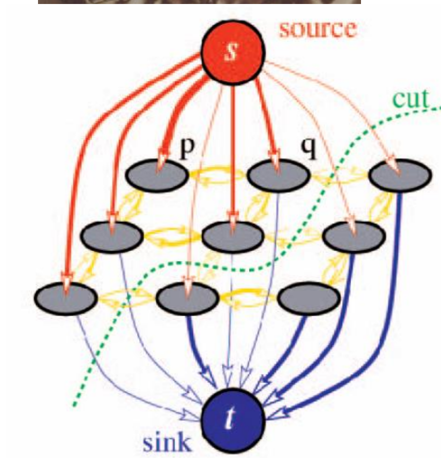
Little noise

noisy

- **Measures of edge and bar strength via derivatives**
- **Ridges in images,**
  towards finding edges and bars



- Interpolation of discrete images
  - Via convolution; via orthogonal basis functions
  - Via splines
  - Via least-squares approximations
- Discrete images as algebraic graphs,
  with objects as graph cuts



Loci as Height Ridges in Graphs

Ref: D. Eberly, *Ridges in Image and Data Analysis*, *Kluwer*

Examples: Edges, Bars

The challenge: identify a point and direction (and for a bar, width) as being on an edge or bar

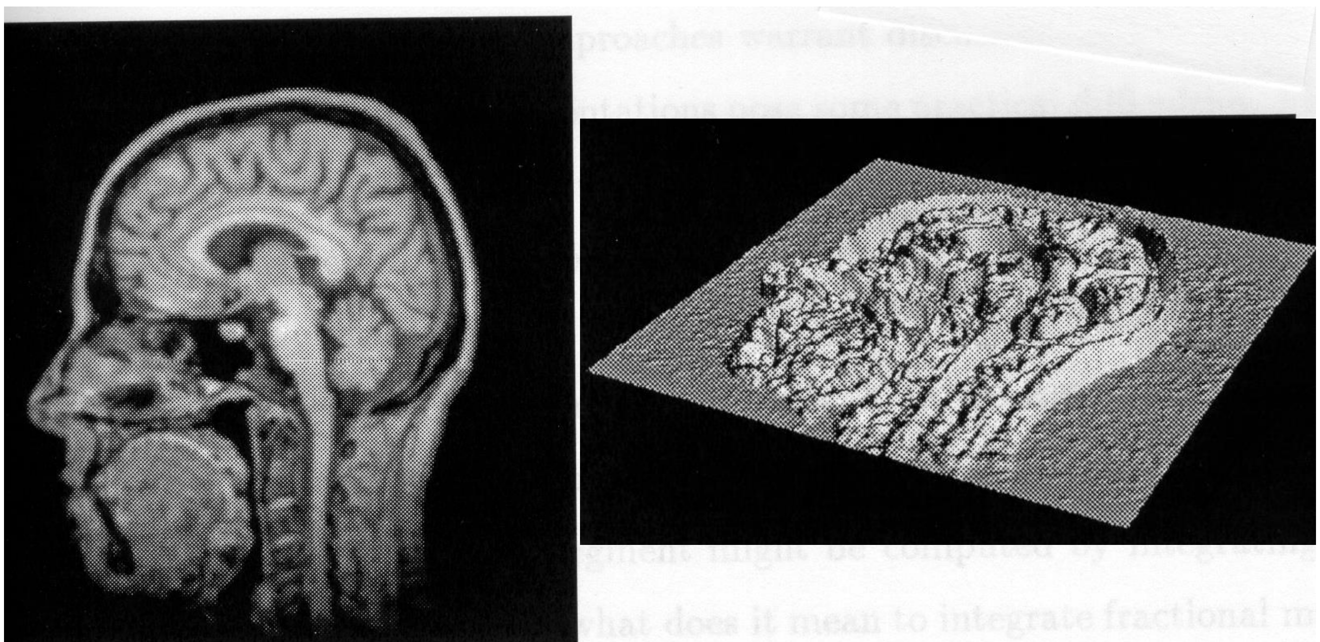
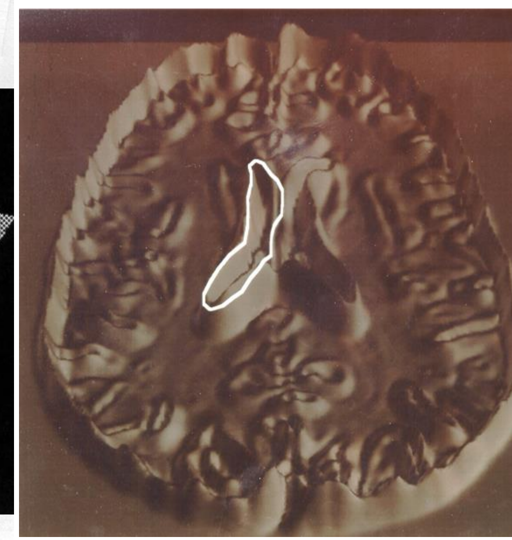


Figure 2.1: The MRI head image has an associated intensity surface.

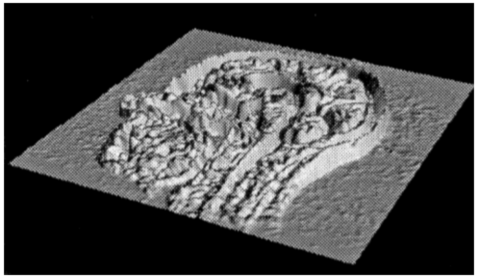


Gradient magnitude

The dimension of a locus in M dimensions

- A smooth locus in M-dimensions has a tangent flat locus
  - A curve (in 2D or 3D or …) has a tangent line
  - A curved surface (in 3D or …) has a tangent plane
  - Etc.
- The dimension, m, of the smooth locus is that of its tangent. Thus,
  - A curve has dimension 1 (in any ambient dimension)
  - A curved surface has dimension 2 (in any ambient dimension)

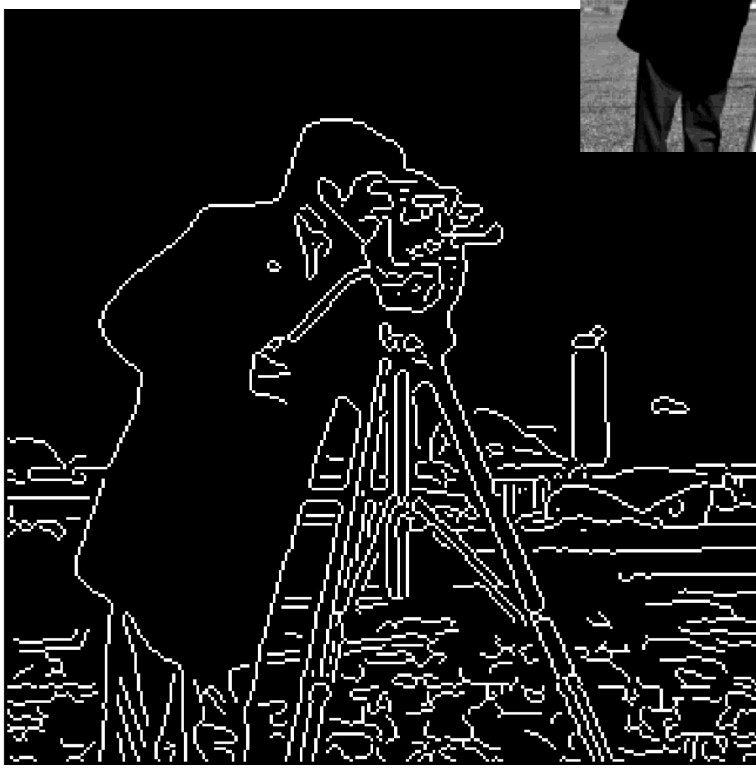
1D "Height" Ridges in f($\underline{z}$) w/ $\underline{z}$ in 2D



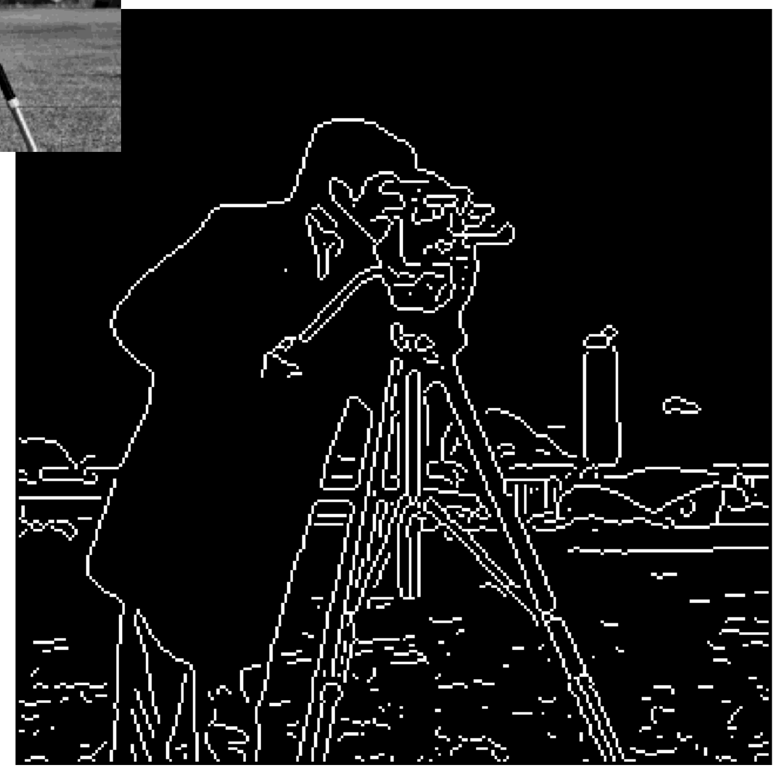
- M=2= dimension of space of $\underline{z}$
  - E.g., $\underline{z}$=$\underline{x}$; 2 = dimension of image
- Ridge is a 1D locus: m=1 in $\underline{z}$
  - To produce a ridge of dimension m, find locus of **relative maximum** is in M-m dimensions
    - M-m=1, so ridge is a 1D relative maximum, in some direction **v** that varies with position $\underline{x}$ on ridge
    - $D^1_{\mathbf{v}} f(\underline{x}) = 0$, $D^2_{\mathbf{vv}} f(\underline{x}) < 0$ (concave downward)
- The issue: what direction **v** ?
  - A common choice: arg $\min_{\mathbf{v}} D^2_{\mathbf{vv}} f$
    - Called maximum convexity ridge: **v**=eigenvector of Hessian
  - In Canny edge, max edge strength direction is made for **v**

Canny Edge Points

Original



Low threshold.



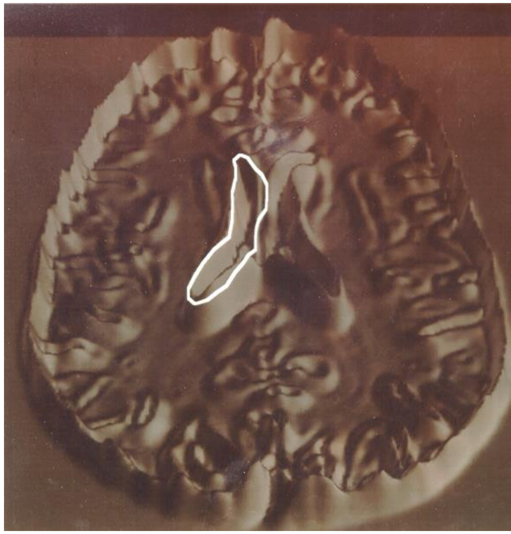
High threshold.

Maximum Convexity 1D "Height" Ridges in 2D

- M=2= dimension of space of $\underline{x}$
  - Height ridge dimension m=1, so max in 1 direction
- Find an initial $\underline{x}$ such that at least one eigenvalue of Hessian D^2J is negative; then march up hill to the ridge
  - Take a step in a direction **v**, chosen as follows
    - Consider eigenvector with most negative eigenvector
    - To get sense to go uphill along **v**, look at $\nabla J(\underline{x})$
  - Take a step along that eigenvector, and retest until $D^1_{\mathbf{v}}J(\underline{x}) = \mathbf{v} \bullet \nabla J(\underline{x})$ changes sign
- Having found a ridge point, take a step $\perp$ to the previous **v** step, and then move back up to ridge by the just described method

1D "Height" Ridges as Edge in 2D

- M=2= dimension of space of $\underline{z}$
  - E.g., $\underline{z}=\underline{x}$; 2 = dimension of image
- Edge is 1D Ridge of $f(\underline{x}) = |\nabla I(\underline{x})|$
- The issue: what direction **v** ?
  - In **Canny edge**, max edge strength direction is taken for **v**
    - $\arg\max_{\mathbf{v}} D^1_{\mathbf{v}} I$ = gradient direction $\nabla I(\underline{x}) / |\nabla I(\underline{x})|$
    - So $\underline{x}$ is a ridge point if $\underline{x} = \arg \text{rel} \max_{\underline{y}} [D_{\nabla I(\underline{y}) / |\nabla I(\underline{y})|} |\nabla I(\underline{y})|]$



2D "Height" Ridges in $f(\underline{z})$ w/ $\underline{z}$ in 3D

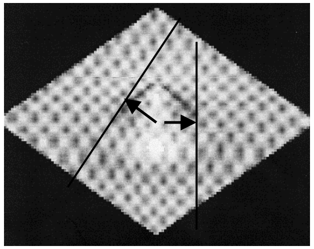
- M=3= dimension of space of $\underline{z}$
  - E.g., $\underline{z}=\underline{x}$; 3 = dimension of image
- Ridge is a 2D locus: m=2
  - M-m=1, so ridge is a 1D relative maximum, in some direction **v**
    - $D^1_{\mathbf{v}}f = 0$, $D^2_{\mathbf{vv}}f < 0$
- The issue: what direction **v** ?
  - A common choice: $\arg\min_{\mathbf{v}} D^2_{\mathbf{vv}}f$
    - Called maximum convexity ridge: **v**=eigenvector of Hessian
    - In Canny edge, $f(\underline{x}) = |\nabla I(\underline{x})|$ and max edge strength direction $\nabla I(\underline{x}) / |\nabla I(\underline{x})|$ is made for **v**

1D "Height" Ridges in f($\underline{x}$,σ) w/ $\underline{x}$ in 2D

- M=3= dimension of space of $\underline{x}$,σ
  - E.g., $\underline{x}$ =dim. of image = 2; edge with its scale
- Ridge is a 1D locus in $\underline{x}$,σ: m=1
  - M-m=2, so ridge is a 2D relative maximum, in some 2D space of directions **v**
    - $D^1_{\mathbf{v}}f = 0$ for 2 orthog dim'ns, $D^2_{\mathbf{vv}}f < 0$ in all dim'
- The issue: what directions **v** ?
  - A common choice: two orthog'l arg $\min_{\mathbf{v}} D^2_{\mathbf{vv}}f$
    - Called maximum convexity ridge: **v**=eigenvectors of Hessian
  - Or Optimal scale ridges in scale space
    - $\mathbf{v}^1$ = pure σ direction; other $\mathbf{v}^i$, are pure space directions, by maximum convexity or optimal parameter

Optimal Parameter Ridges

- A geometrically defined direction for which f is optimum
  - E.g., for edge: $f = |\nabla I|$; $\mathbf{u}^1 = \nabla I$ direction, i.e., direction for which f is a relative maximum
    - Called the Canny edge criterion
  - E.g., for bar: $f = \nabla^2 G(\underline{x};\sigma) * I(\underline{x})$; $\mathbf{u}^1 = \mathbf{v}^1 - \mathbf{v}^2$ with each $\mathbf{v}^i$ = directions of rel max of contribution to f
    - Specified σ or optimal σ



Barness in Position, Width, Orientation in a 2D image – An Alternative*

- **-** $\mathbf{v}^T D^2 f(\underline{x}, \sigma)\, \mathbf{v}$ vs. $\underline{x}, \sigma, \mathbf{v}$ (4D)
- Locus gives bar position (orthogonal to the bar: 1 dimension in x,y; width (1 dimension: σ); and direction (orthogonal to the bar: 1 dimension: **v**)
  - Thus a 1D ridge in 4D
  - 1D Locus of high values forms a ridge
  - Directions of maximization
    - **v** (orientation)
    - spatial position along **v**
    - σ
  - **-**D^2 in the subdimensional 3D must have negative eigenvalues (convex downward)

"Height" Ridges in f($\underline{z}$)

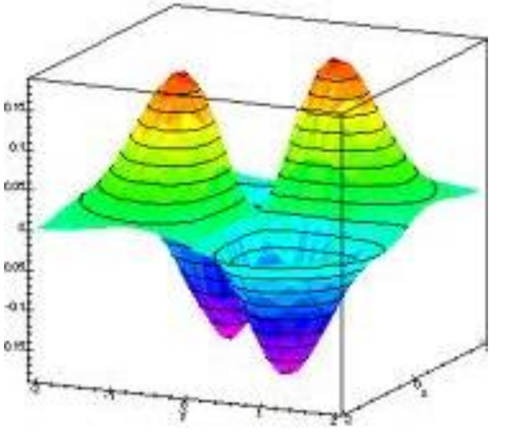
- M= dimension of space of $\underline{z}$
  - E.g., $\underline{z}=\underline{x}$; M = dimension of image
  - E.g., scale space: $\underline{z}=\underline{x},\sigma$; M = 1+ dim of image
  - E.g., scale space and direction: $\underline{z}=\underline{x},\sigma,\underline{\theta}$; M = dim of image + 1 + dim of image -1
    - Last term gives number of angles to determine a direction, i.e., a point on sphere of dimension the same as the image
- Ridge is a subdimensional relative maximum
  - To produce a ridge of dimension m, the relative maximum is in M-m dimensions
    - When m=0, the ridge is a relative maximum

When Ridges End: Valleys and Connectors

- Maximum-convexity ridge: $(M\text{-}m)^{th}$ most negative eigenvalue of $D^2f(\underline{z}_0)$ crosses zero
  - $D^2f(\underline{z}_0)$ becomes singular
  - $\mathbf{u}^{M\text{-}mT} D^1f(\underline{z}_0)$ still = 0, since level curve cannot end, but graph becomes flat then convex upward in direction $\mathbf{u}^{M\text{-}m}$
  - So-called “connector
  - Later, can recross zero to become ridge or another eigenvalue can cross zero
    - In 2D that corresponds to being a valley point

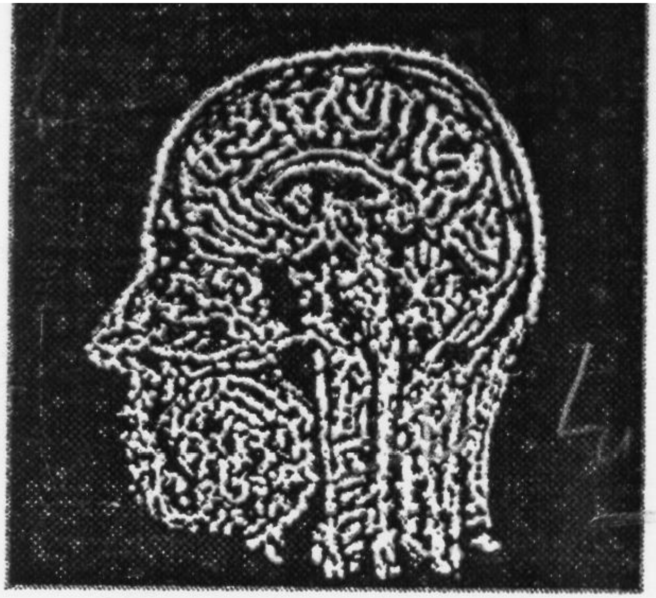
Ridge Types Other than Height Ridges

- Watershed ridges
  - Divide regions where going downhill arrives at common relative minimum
    - Split at passes
  - Will not catch all height ridges
  - Unlike height ridges, non-local
  - There are fast algorithms to compute them

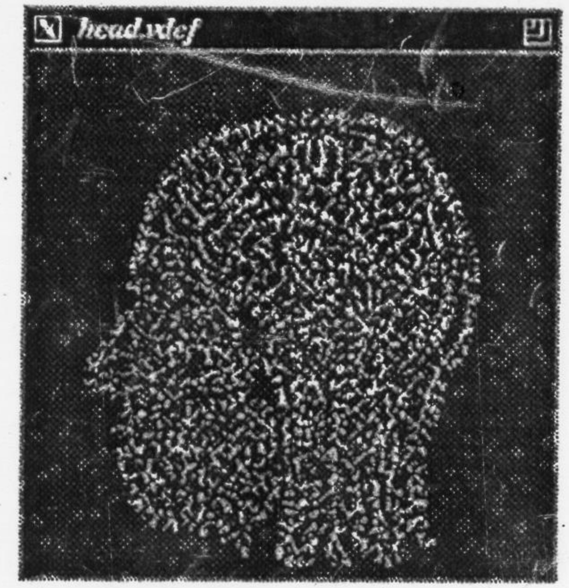


- Vertices of level curves
- Crests: ridges of surfaces, not of graphs

Ridge Types Other than Height Ridges

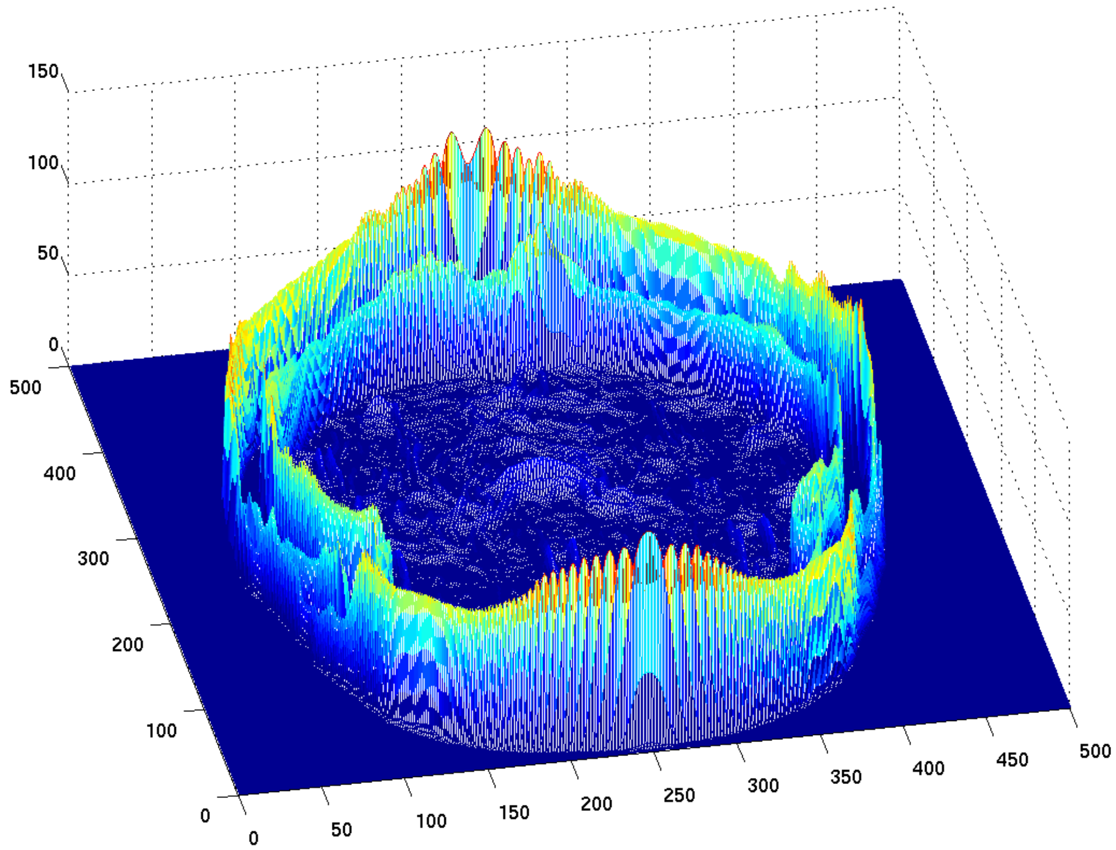


Local maxima of height in eigendirections



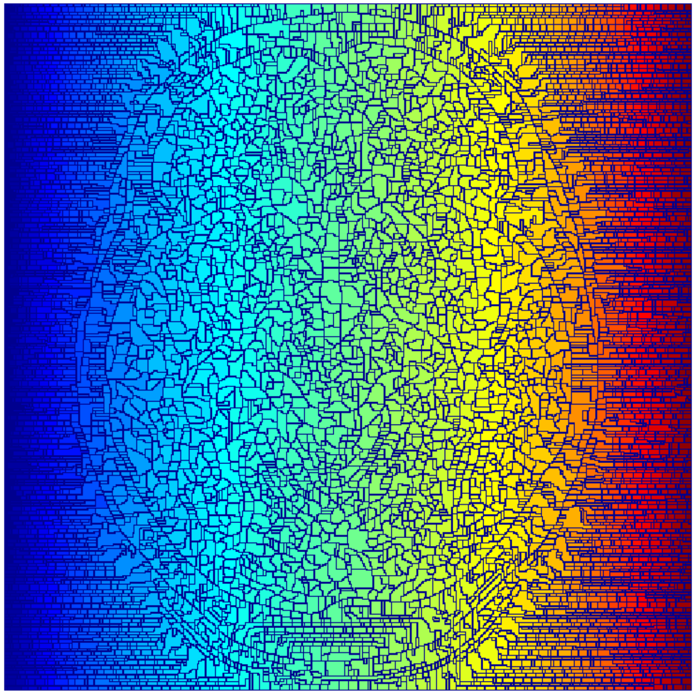
Local maxima of curvature along level sets

Watershed

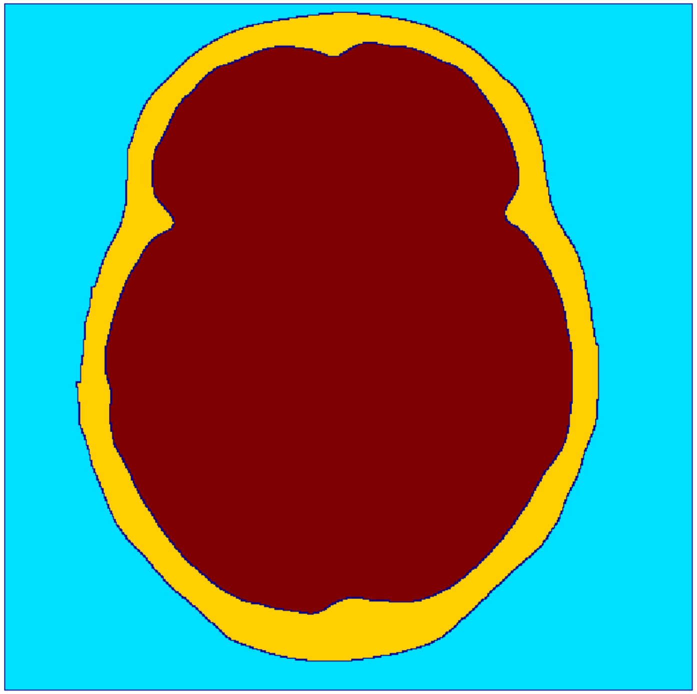


Gradient magnitude image interpreted as height map.

Watershed



Direct application leads to oversegmentation.



With merging (minimum depth)

Ref: Ole Fogh Olsen & Mads Nielsen, Multi-scale gradient magnitude watershed segmentation, Proc. ICIAP, Springer LNCS: Vol. 1310, 1997