The Math Needed to Understand Image Processing cont.

- Two aspects of spatial scale
  - Levels of detail
  - Gaussian apertures and spatial scale
- Intensity noise vs. scale
- Measures of edge and bar strength via derivatives
- Ridges in images, towards finding edges and bars
- Interpolation of discrete images
  - Via convolution; via orthogonal basis functions
  - Via splines
  - Via least-squares approximations
- Discrete images as algebraic graphs, with objects as graph cuts
Scale and Locality

• Two different factors called *spatial scale* of a sample or a basis function
  – level of detail: basis functions $\psi^{lod}(u)$
    • So 1 basis function per lod;
      e.g., $lod = \text{sinusoid wavelength}$
  – aperture (with locality): $\psi(u, u_0, \sigma; lod)$,
    • Involves an aperture weighting function centered at a location $u_0$
    • Determines interrelation distances,
      e.g., bar or disk widths
    • A whole set of basis functions at each scale
  – Both factors determine feature size on which to focus
Scale and Locality

• Birchfield on aperture size scale aspect
  – Section 7.1
• Birchfield on Laplacian of Gaussian
  – Section 5.4.1
• Birchfield on Gaussian derivative kernels
  – Section 5.3
• Birchfield on the Gaussian kernel
  – Section 5.2
Focusing on the right scale

• Example: white noise in a blurred image
  \[ I_{\text{discrete}}(x,y) = I_{\text{discrete \& ideal}}(x,y) + \text{noise}(x,y) \]

• Choose aperture size to delete or attenuate undesired scales

• Choose level of detail to focus on lods with good signal-to-noise, i.e., large and moderate lods

• Example: remove unwanted detail

• Example: bar or blob width
Apertures and Levels of Detail

- Apertures
  - Global
  - Local
    - Gaussian and its derivatives
      - Aperture scale is $\sigma$
    - Splines
      - Aperture scale is size of data support for a patch
    - With orthogonality:
      - Gabor functions: sinusoid under Gaussian with aperture scale $\sigma$
      - Orthogonal wavelets

- Levels of detail
  - Sinusoid wavelength (also for Gabor)
  - Derivative order
  - Spline data grid spacing
  - Orthogonal wavelet binary decimation level
Properties desired of an aperture

• Unbiased re spatial scaling, translation, rotation
• Cascading apertures gives a legal aperture
• Do not create structure, only eliminate it
• Have finite integral

• The only continuous aperture that does all of that is the Gaussian!
  • Book exists as a Mathematica program (can chg the figures)
Non-creation of structure

- No new level curves very nearby
  - Of intensity
  - Of derivatives of intensity

- Equivalently, upon application of aperture
  - local maxima disappear or decrease in intensity
  - local minima disappear or increase in intensity

- Not equivalent to no creation of local maxima or minima
  - Consider taut curtain between mountain-tops
Properties of the Gaussian

- Separable
- Convolution or product of 2 Gaussians is Gaussian
- Rotationally invariant; i.e., isotropic
  - Also ellipsoidal form is available
- Is its own Fourier transform, but reciprocal std deviation
- Central limit theorem: \( \ast \sum_{i=1}^{n} h_i \) is Gaussian in the limit
- Maximum entropy: most uncertain with fixed variance
- Diffusion (heat equation): \( \frac{\partial f(x,t)}{\partial t} = \nabla^2 f \) with \( f(x,0)=\delta(x) \) has Gaussian as solution (psf, convolution kernel)
- Scale (apertures) that
  - are agnostic re rotation, translation, and magnification
  - compose successive scale changes into a single scale change
  - do not create structure by increasing scale
- Result of Brownian motion
The Gaussian

- Formula:
  - Isotropic: \((2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2}[|x-\mu|/\sigma]^2\}\)
  - General: \((2\pi)^{-n/2} |\text{det } \Sigma|^{-1/2} \exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}\)

- Eigenvectors of \(\Sigma\) are principal directions
- \((\text{Eigenvalues of } \Sigma)^{1/2} \propto \text{principal radii}\)
White Intensity Noise vs. Image

- If the noise in each pixel is uncorrelated with the noise in any other pixel, the noise is called white.
  - Called white because for orthogonal basis images the expected square value of each basis image coefficient of noise instances is constant, i.e., does not vary with lod.
- Scenes tend to be close to white, i.e., over a population have the same average squared orthogonal basis image coefficients.
- Images are scenes $\ast$ psf of camera, and transfer function magnitude falls with frequency.
- The signal (image) to noise falls as lod falls, eventually having noise dominate past some lod.
  - Implies amplification of basis function coefficients as lod decreases (e.g., frequency increases) needs to turn into damping below some lod.
How to Compute a First Derivative of M-D Image

• Always via derivative of Gaussian
  – Will damp lod components below some threshold

• In a non-cardinal direction
  – Compute the M cardinal derivatives in the gradient
    • If done via freq. domain (see below), multiply amplitudes (or both real and imaginary parts) by M-D Gaussian once
  – Dot product result with direction

• In a cardinal direction, say x
  – If Gaussian’s $\sigma < 3$ pixels, operate in space domain
    • Compute Gaussian kernel and apply that narrow (<8 pixels wide) weighting function pixel by pixel
  – Otherwise, take FFT of image and operate in frequency domain
    • Multiply Gaussian-updated amplitudes (or real and imaginary parts) by $\nu_x$, and if necessary by the $2\pi$ that is part of $2\pi i$
    • To effect multiplication by $i = e^{i\pi/2}$, add $\pi/2$ to every phase in FFT(I) (or change sign of imaginary party of FFT(I) and then swap real and imaginary parts of the result

• If in another cardinal dir., say y, only change $\nu_x$ to $\nu_y$
Splines

- Smoothly connected patches
- Typically a polynomial in each patch
- Local support by nearby grid elements
The Part of the Course Covered on the Midterm Ends Here
Scale Situations in Various Sampled Geometric Analysis Approaches

<table>
<thead>
<tr>
<th>Global coef for each level of detail</th>
<th>Multidetail feature</th>
<th>Detail residues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Examples: Fourier coeffs, global principal components</td>
<td>image pixels boundary points, dense displacements</td>
<td>orthogonal wavelets Gabor, Gauss deriv, recursive splines</td>
</tr>
</tbody>
</table>

![Diagram showing scale situations in various sampled geometric analysis approaches.](image)
Polynomial Basis Functions w/ Locality

• Splines: patchwise fitting
  – Approximating
    • E.g., B-splines, related to wavelets

\[
Q_i(t) = \frac{1}{6} [(t-t_i)^3 (t-t_i)^2 (t-t_i) 1]
\]

\[
\begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
P_{i-1} \\
P_i \\
P_{i+1} \\
P_{i+2}
\end{bmatrix}
\]

• Above is in each dimension; separable
• \( t_i \) integers, (\( t-t_i \) \( \in [0,1] \))
  – Interpolating: Global
  – Approximating: Locality (aperture) by control point (\( P_i \)) spacing
Some Uses of Splines

• Smooth bias fields for images
  – Subtract it

• Smooth sensitivity fields for images
  – Divide by it

• Smooth displacement fields for distortion
  – Separate splines for $\Delta x(x), \Delta y(x), \Delta z(x)$
  – Also used to compute deformable registrations
    • Optimize $\Delta x$ control point sets
Representations with Locality

- With parametrized representation
  - With \( u \) as parameter(s)
  - With \( f = x \) or \( I \) or … as function of \( u \)

- Need \( f(u, \sigma) = \sum_{\text{lod}} a(\text{lod}, u_0, \sigma) \psi^{\text{lod}}(u, u_0, \sigma) \)
  - \( u \) is location
  - \( \sigma \) is aperture size (typically std dev of Gaussian), \( u_0 \) is aperture center
  - \( \text{lod} \) is level of detail
    - For Fourier it is wavelength \( 1/\nu \) for frequency \( \nu \)
    - For derivative, it is order
      - Not too noisy if \( \sigma \) is well chosen and order < 6
    - For orthogonal wavelet it is level of decimation
References for Representations with Locality

• B-splines
  – Birchfield, section 4.6.5
  – E. Cohen, R. Riesenfeld, G. Elber. *Geometric Modeling with Splines*

• Gabor wavelets
  – Birchfield, section 6.6.7

• Orthogonal wavelets
  – Birchfield, section 6.6.5
Basis Functions with Locality

• Gabor functions: sinusoids under the Gaussian [ref Wechsler: *Comp’l Vision*]
  – Like derivatives of Gaussian, with $\nu \propto$ derivative order
  – Wavelength $1/\nu \propto \sigma$
  – Sampling $\propto \sigma$

• Orthogonal wavelets
  – Interscale residues
  – Orthogonality across & within scales

† The 20th order Gaussian derivative
Separability vs. Isotropy with local basis images

• Separability means apply in one coordinate, then in the other on the result
• Isotropy implies lack of bias toward orientation
• Gabor functions applied separably and derivatives of Gaussians applied separably are isotropic
• Splines and orthogonal wavelets are applied separably but that does not produce isotropy
Basis Functions with Locality: Derivatives of Gaussian

- [ref: ter Haar Romeny book]
- Order of derivative is LOD
  - Not orthogonal as basis functions
  - Sampling $\propto \sigma$ (see later slide on derivatives)
- Not too different in effect from Gabor wavelets
- Localized Taylor series
- Diffusion equation and Taylor series in $t (=\sigma^2/2)$ yields Laplacian-based local multi-scale approximation
Pyramids: Images in Scale Space

- Gabor $\Delta_\sigma I = G(x;\sigma)*I(x) - G(x;2\sigma)*I(x)$
  $= [G(x;\sigma) - G(x;2\sigma)]*I(x)$
  - As scale $\sigma$ increases, the sampling distance can increase proportionally

- $G(x;\sigma) - G(x;2\sigma) \approx \nabla^2 G(x;\sigma)$

- As you increase scale by some constant factor, you produce an image as a function of $x$ (with adjusted sampling) and $\sigma$: the Laplacian pyramid
  - Small scale to large

- If you combine the Laplacian effect with the orthogonal wavelet, you get orthogonal wavelet pyramid
Uses of basis functions with locality

• Three dimensions
  – Location
  – Aperture size
  – LOD, often best $\propto$ aperture size

• Choosing the aperture size and LOD
  – PCA (SVD)
  – Biggest average response(s) per location

• Edgeness and barness operators
  – Edgeness: directional 1$^{st}$ derivative with appropriate aperture: cf. edge slope
  – Barness: directional 2$^{nd}$ derivative with appropriate aperture: cf. bar width
Loci as Height Ridges in Graphs
Ref: D. Eberly, *Ridges in Image and Data Analysis*, Kluwer

Examples: Edges, Bars (ridges will come in next course section)

The challenge: identify a point and direction (and for a bar, width) as being on an edge or bar

Figure 2.1: The MRI head image has an associated intensity surface.
Edgeness and barness operators

• Edgneness
  – Gradient with aperture: \( D^1 f(x, \sigma) = \nabla f(x, \sigma) = \left[ \frac{\partial G(x, \sigma)}{\partial x_1} f, \ldots, \frac{\partial G(x, \sigma)}{\partial x_n} f \right]^T \)
    • Gives direction of maximum edgness
    • Magnitude gives amount of edgeness in that dir.
  – Directional derivative with aperture = edgeness in the \( v \) direction = \( v \cdot \nabla f(x, \sigma) \)

• Barness
  – Hessian with aperture: \( D^2 f(x, \sigma) = M \times M \) matrix
    \[ \left[ \frac{\partial^2 G(x, \sigma)}{\partial x_i \partial x_j} \right] \]
  – Barness: directional 2\(^{nd}\) derivative w/ appropriate aperture in the \( v \) direction = \( -v^T D^2 f(x, \sigma) v \)
Optimal Barness Direction

- $\text{Max}_v [- v^T D^2 f(x, \sigma) v ]$
  - $D^2 f$ is symmetric
  - Thus $v = \text{eigenvector of } D^2 f$ with most negative eigenvalue
  - Thus, barness is the negative of the most negative eigenvalue of $D^2 f$
Edgeness and Barness Seen as Matched Filters

- Edgness via $\nabla f(x, \sigma) = v \cdot \nabla f(x, \sigma)$ with $v$ being unit vector in gradient direction.

- $\text{Max}_{|v|=1}[- v^T D^2 f(x, \sigma) v]$ attained with $v$ being unit eigenvector with most negative eigenvalue.

- Kernel corresponding locally to edge or bar respectively matches the edge itself.
  - $\text{Max}_{|h|=1} h(x)^* q(x)|_{x=0}$ attained when $h(x) = q(x)/|q(x)|$.
Aperture size for edgeness and barness operators (and other derivatives)

• Derivatives are not commensurable
  – 1\textsuperscript{st} derivatives have units of intensity/mm
  – 2\textsuperscript{nd} derivatives have units of intensity/mm\(^2\)
  – Etc.

• Make them commensurable by multiplying \(k\textsuperscript{th}\) derivative by aperture’s \(\sigma^k\)
  – (or \((c\sigma)^k\))
  – Make them comparable via error propagation behavior (see next slide)
Error Propagation of $\sigma^k$-Scaled Derivatives

- 1D
- Relative to error in $0^{th}$ derivative
- Displayed as log (so at order 0, is zero)
Error Propagation under Convolution with Gaussian

• Noise level (standard deviation) is multiplied by $[\int h^2(x) \, dx]^{1/2}$, with $h$ the Gaussian
  – That is, in $M$ dimensions output noise level is divided by $\sigma^{M/2}$

• Thus change propagation by choosing $\sigma$ for each derivative order to achieve the desired level of propagation
White Intensity Noise vs. Level of Detail

- \( I_{\text{noisy}}(x) = I_{\text{noise-free}}(x) + \text{noise}(x) \)
- In any orthonormal function basis, noise that is uncorrelated between pixels has constant variance in every basis function coefficient.
- In that basis, as lod increases, the coefficient \( a_{\text{noise-free}}(\text{lod}) \) of \( I_{\text{noise-free}}(x) \) roughly falls like a Gaussian.
- Thus signal-to-noise = \( a_{\text{noise-free}}(\text{lod}) / \sqrt{\text{var}(a(\text{lod}))} \) falls as lod increases, eventually becoming \(< 1\).
Uses of Spatial Scale

- Measurements at an appropriate chosen scale
- Optimality in scale space: best scale at each location
- Decomposition into residues at various scales

LOD

One orthonormal basis function coefficient per box location
## Comparative Properties of Main Parametrized Decompositions

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<tr>
<th></th>
<th>Locality</th>
<th>Invariances</th>
<th>Speed</th>
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<tbody>
<tr>
<td><strong>Fourier</strong></td>
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<td><strong>Gaussian</strong></td>
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