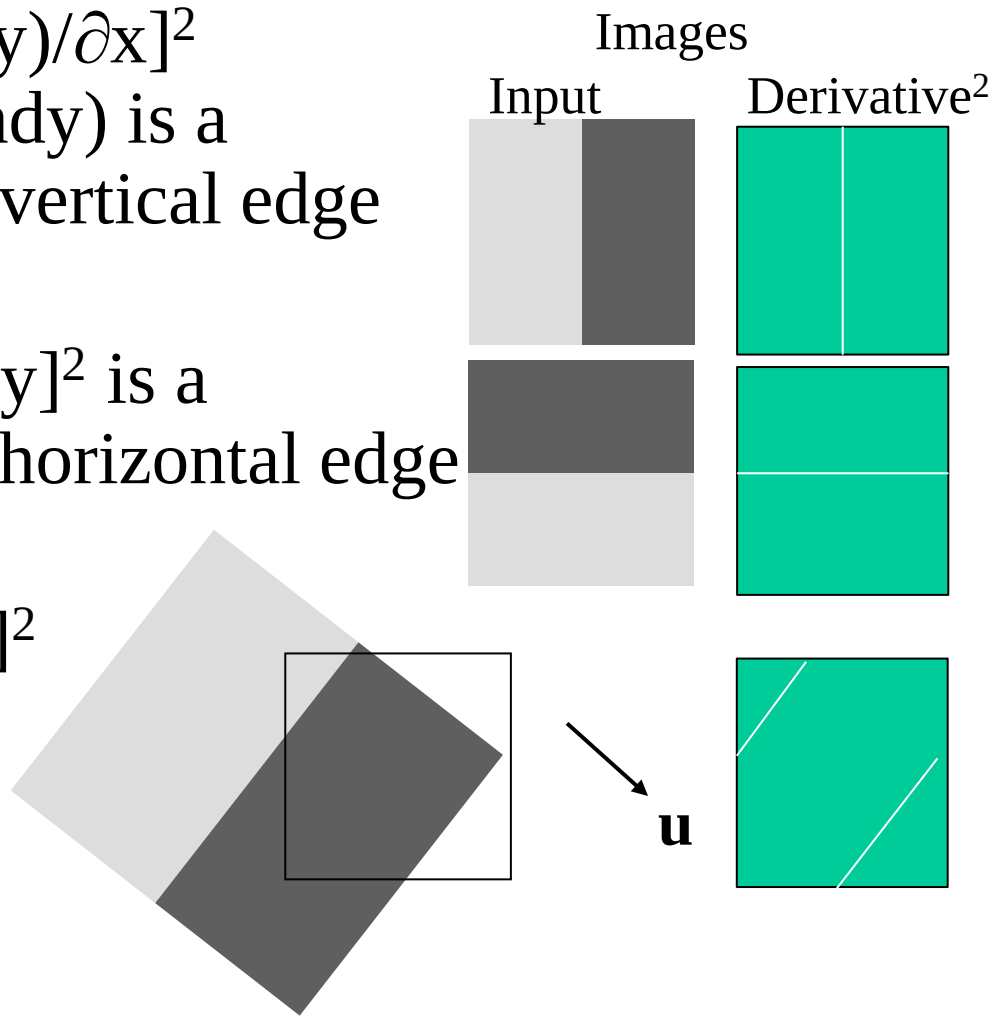
The Math Needed to Understand Image Processing

- **Representation of images as Taylor series**
  - **Thus computation of image derivatives**
- Invariant operators: to shift, rotation, scale
  - T is invariant to G if for all images I, $T(G(I))=G(T(I))$
- Shift-invariant, linear operators: G = translation
  - Linear operators T have property that for all images, I, J and scalars α, β,
    - $T(\alpha I+\beta J) = \alpha T(I)+\beta T(J)$
    - Or equivalently, $T(I+J) =T(I)+T(J)$ and $T(\alpha I) = \alpha T(I)$
      - We will see that such operators have a close connection with convolution
  - <u>All</u> derivatives are linear and shift-invariant
    - By all derivatives, the following are included
      - All orders, directional derivatives, partial derivatives, the Laplacian
        - » The Laplacian $\nabla^2 I= \Sigma_{i=1}^{\#\,dims}\, \partial^2 I/ \partial x_i^2$ for any orthogonal coord sys

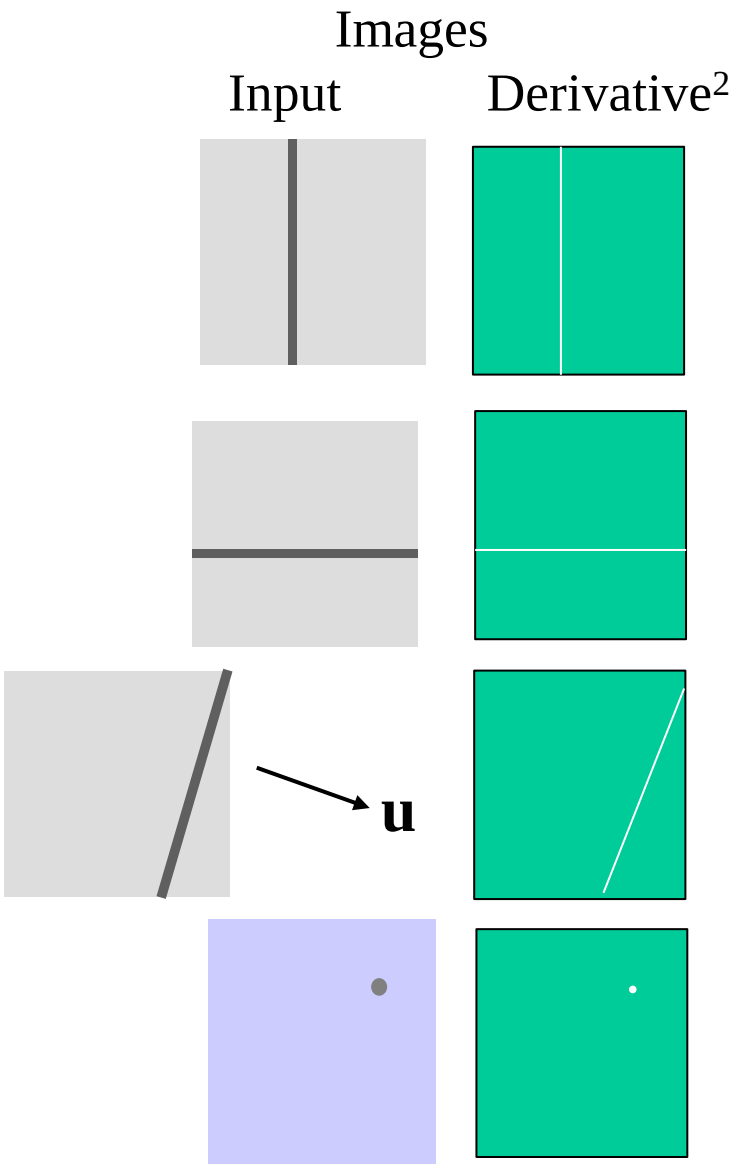
Spatial Derivatives and Edges

- First derivatives (w.r.t. space) are important indicators of edges and bars
  - The 2D image $[\partial I(x,y)/\partial x]^2$ (as a function of x andy) is a strong indicator of a vertical edge
  - The image $[\partial I(x,y)/\partial y]^2$ is a strong indicator of a horizontal edge
  - The image $[D_{\mathbf{u}} I(x,y)]^2$ is a strong indicator of an edge with normal direction **u**



Spatial Derivatives and Bars and Blobs

- Second derivatives (w.r.t. space) are important indicators of narrow bars
  - The image $[\partial^2 I(x,y)/\partial x^2]^2$ is a strong indicator of the center of a narrow vertical bar
  - The image $[\partial^2 I(x,y)/\partial y^2]^2$ is a strong indicator of a narrow horizontal bar
  - The image $[D^2_{\mathbf{uu}} I(x,y)]^2$ is a strong indicator of a narrow bar with edge-normal direction **u**
- $|\nabla^2 I|^2$ is a strong indicator of a circular blob



Spatial Derivatives via Tensors

- **Every directional spatial derivative of order k at $\underline{x}_0$ of $f(\underline{x})$ with $\underline{x} \in \Re^n$ is captured by a kth order symmetric tensor with the entry with index $i_1 i_2 \ldots i_k$ being given by $\partial^k f / \partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}(\underline{x}_0)$**
  - x_j is the jth of the scalar variables in $\underline{x}$
- A kth order tensor $\in \Re^n$ is an $n \times n \times \ldots \times n$ (k times) array
  - A zero order tensor is a 1-entry (1×1) array
  - A first order tensor is an $n \times 1$ array (column vector)
  - A second order tensor is an $n \times n$ array (matrix)
  - A third order tensor is an $n \times n \times n$ array
  - Etc.
- Pre- (post-) multiplying a kth order tensor by a row (column) vector yields a $(k-1)^{th}$ order tensor obtained by taking the dot-product between that vector and each of the tensor columns with first (last) index
  - For example if k=1, the result is a 1-entry (0th order) tensor obtained by taking the dot product of the vector and the column vector forming the 1-tensor
  - For example if k=2, the result of a pre-multiplication by a vector is a 1-tensor (vector) with its ith element obtained by taking the dot product of the vector and the ith column vector of the 2-tensor (matrix); and the result of a post-multiplication by a vector is a 1-tensor (vector) with its ith element obtained by taking the dot product of the vector and the ith row vector of the 2-tensor (matrix)

Taylor Series and Spatial Derivatives

- Taylor series express an image **with local accuracy** as a polynomial whose coefficients are derivatives normalized by factorials
  - In 1D, $I(x) \approx I(x_0) + \Sigma_{k=0}^{n} [d^kI/dx^k(x_0) / k!] (x-x_0)^k$
- For an image of arbitrary dimension M, e.g., 2D (M=2) or 3D (M=3), all k^{th} derivative values at a point $\underline{x}_0$ are captured by the k-tensor $D^kI(\underline{x}_0)$, a linear operator represented by an array of size $M \times M \times \ldots \times M$ (with dimension k) with the array element $i_1, i_2, \ldots, i_k$ being $\partial^kI/ \partial x_{i_1}\partial x_{i_2} \ldots \partial x_{i_k} (\underline{x}_0)$
  - Examples:
    - for k=1, $D^1I(\underline{x}_0)$ is the gradient vector with entries $\partial I/\partial x_i$
    - for k=2, $D^2I(\underline{x}_0)$ is the Hessian matrix with entries $\partial^2I/ \partial x_{i_1}\partial x_{i_2}$
- The kth directional derivative at $\underline{x}_0$ w.r.t. to the directions $\mathbf{U} = \mathbf{u}_{i_1} \mathbf{u}_{i_2} \ldots \mathbf{u}_{i_k}$ is a scalar obtained by the successive linear operations of vector-product of $D^kI(\underline{x}_0)$ by $\mathbf{u}_{i_j}$ by the k-tensor $D^kI(\underline{x}_0)$, j = 1, 2, …, k (or in any other order)
  - Call this operation $\mathbf{U} \bullet D^kI(\underline{x}_0)$
  - The order of the operations does not matter; all the operations but the last are on the left (mult. by $\mathbf{u}^T$), but for k>1 the last must be on the right (mult. by $\mathbf{u}$)

Taylor Series and Spatial Derivatives

- Taylor series express an image **locally** as a polynomial whose coefficients are derivatives normalized by factorials
  - In 1D $I(x) \approx I(x_0) + \Sigma_{k=1}^{n} [d^k I/dx^k(x_0) / k!] (x-x_0)^k$
- For an image of arbitrary dimension M, e.g., 2D (M=2) or 3D (M=3), the Taylor series approximation for $I(\underline{x})$ with $\underline{x}$ near $\underline{x}_0$ is

$$I(\underline{x}) \approx I(\underline{x}_0) +$$
$$\Sigma_{k=1}^{n} \Sigma_{\underline{i} = \text{all } M^k \text{ choices of } i_1, i_2, \ldots, i_k} [\mathbf{U}_{\underline{i}} \bullet D^k I(\underline{x}_0) / k!] \times$$
$$\Pi_{j=1}^{k} (x_{i_j} - x_{0_{i_j}}),$$

where each i_m is chosen from 1, 2, …, M, $\Sigma_{\underline{i}=1}^{n} i_j = k$,

and $\mathbf{U}_{\underline{i}} = \mathbf{u}_{i_1} \mathbf{u}_{i_2} \ldots \mathbf{u}_{i_k}$

Taylor Series and Spatial Derivatives

- Taylor series express an image **locally** as a polynomial whose coefficients are derivatives normalized by factorials
  - Alternative form to

    $I(\underline{x}) \approx I(\underline{x}_0) +$

    $\Sigma_{k=0}^{n} \Sigma_{\underline{j} = \text{all } M^k \text{ choices of } i_1, i_2, \ldots, i_k} [\mathbf{U}_{\underline{j}} \bullet D^k I(\underline{x}_0) / k!] \times \Pi_{j=1}^{k} (x_{i_j} - x_{0_{i_j}})$,

    where each i_m is chosen from 1, 2, …, k

    and $\mathbf{U}_{\underline{j}} = \mathbf{u}_{i_1} \mathbf{u}_{i_2} \ldots \mathbf{u}_{i_k}$ is as follows:
  - **[This is the form we actually use.]** Consider the vector $(\underline{x}-\underline{x}_0)$.

    The Taylor series approximation for $I(\underline{x})$ with $\underline{x}$ near $\underline{x}_0$ is

    $I(\underline{x}) \approx I(\underline{x}_0) + \Sigma_{k=0}^{n} \text{term}_k / k!$, where

    $\text{term}_k = (\underline{x}-\underline{x}_0)^T \bullet [(\underline{x}-\underline{x}_0)^T \bullet \ldots [(\underline{x}-\underline{x}_0)^T \bullet D^k I(\underline{x}_0)] \ldots] \bullet (\underline{x}-\underline{x}_0)$, with the series of vector products including k-1 entries
  - $\Pi_{j=1}^{k} (x_{i_j} - x_{0_{i_j}})$ are **basis images**; $\mathbf{U}_{\underline{j}} \bullet D^k I(\underline{x}_0) / k!]$ are coefficients

Representation of 2D and 3D images as Taylor series

- Thus if we can compute image derivatives,

$$I(\underline{x}) = I(\underline{x}_0) + \frac{1}{1!}\sum_{i=1}^{M}(x_i - x_{0i})\frac{\partial I}{\partial x_i}(\underline{x}_0) + \frac{1}{2!}\sum_{i,j=1}^{M}(x_i - x_{0i})(x_j - x_{0j})\frac{\partial^2 I}{\partial x_i \partial x_j}(\underline{x}_0) + \ldots$$

$$I(\underline{x}) = I(\underline{x}_0) + \frac{1}{1!}(\underline{x} - \underline{x}_0)^T D^1 I(\underline{x}_0) + \frac{1}{2!}(\underline{x} - \underline{x}_0)^T D^2 I(\underline{x}_0)\,(\underline{x} - \underline{x}_0) + \ldots$$

  - where $D^1 I = \nabla I$, the gradient of I, an M-vector with components $\frac{\partial I}{\partial x_i}$

$D^2 I(\underline{x}_0) =$ the "Hessian" of I, an M × M matrix with components $\frac{\partial^2 I}{\partial x_i \partial x_j}$

  - Each image derivative is itself an image
- When you cut off the series to a given number of terms, the result is a continuous image with pixel values computable at any value of $\underline{x}$

Usefulness of Taylor Series and Spatial Derivatives

- Taylor series express an image **locally** as a polynomial in the components of $\underline{x}$ whose coefficients are derivatives normalized by factorials
  - They provide a very poor approximation with a reasonably small number of terms except extremely locally
- The spatial derivatives of order 1 are used to compute edgeness, and those of order 2 are used to compute barness

Exercise
Due as a digital picture at class time on Thurs. 1/20

- Example: assume you are given the 2D image of 2-tensors $D^2 I(\underline{x})$
  - Reminder: $D^2 I(\underline{x})_{ij} = \frac{\partial^2 I}{\partial x_i \partial x_j}(\underline{x}), i = 1,2;\ j = 1,2$

    Given two unit (direction) vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ and $D^2 I(\underline{x})$, a) give a formula for the (scalar) image $D^2 I_{u_1 u_2}(\underline{x})$
    Answer: $u_1^T D^2 I(\underline{x}) u_2$
    |
    b) give a formula for $\frac{\partial^2 I}{\partial y^2}(\underline{x})$ Answer: $e_y^T D^2 I(\underline{x}) e_y$, where $e_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; thus $D^2 I(\underline{x})_{22}$
- Exercise: Given three unit (direction) vectors $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$ and the derivative tensor needed,

a) give a formula for the (scalar) image $D^3 I_{u_1 u_2 u_3}(\underline{x})$

b)) give a formula for the scalar image $\frac{\partial^3 I}{\partial x^2 \partial y}(\underline{x})$