The Math Needed to Understand Image Processing

• Representation of images as Taylor series
  – Thus computation of image derivatives

• Invariant operators: to shift, rotation, scale
  – T is invariant to G if for all images I, T(G(I))=G(T(I))

• Shift-invariant, linear operators: G = translation
  – Linear operators T have property that for all images, I, J and scalars α, β,
    • T(αI+βJ) = αT(I)+βT(J)
    • Or equivalently, T(I+J) = T(I)+T(J) and T(αI) = αT(I)
    – We will see that such operators have a close connection with convolution
  – **All derivatives are linear and shift-invariant**
    • By all derivatives, the following are included
      – All orders, directional derivatives, partial derivatives, the Laplacian
      » The Laplacian \( \nabla^2 I = \sum_{i=1}^{\# \text{dims}} \frac{\partial^2 I}{\partial x_i^2} \) for any orthogonal coord sys
Spatial Derivatives and Edges

- First derivatives (w.r.t. space) are important indicators of edges and bars
  - The 2D image \( [\partial I(x,y)/\partial x]^2 \) (as a function of x and y) is a strong indicator of a vertical edge
  - The image \( [\partial I(x,y)/\partial y]^2 \) is a strong indicator of a horizontal edge
  - The image \( [D_u I(x,y)]^2 \) is a strong indicator of an edge with normal direction \( u \)
Spatial Derivatives and Bars and Blobs

- Second derivatives (w.r.t. space) are important indicators of narrow bars
  - The image \( \left[ \frac{\partial^2 I(x,y)}{\partial x^2} \right]^2 \) is a strong indicator of the center of a narrow vertical bar
  - The image \( \left[ \frac{\partial^2 I(x,y)}{\partial y^2} \right]^2 \) is a strong indicator of a narrow horizontal bar
  - The image \( \left[ D^2_{uu} I(x,y) \right]^2 \) is a strong indicator of a narrow bar with edge-normal direction \( u \)
- \( |\nabla^2 I|^2 \) is a strong indicator of a circular blob
Spatial Derivatives via Tensors

- Every directional spatial derivative of order $k$ at $\mathbf{x}_0$ of $f(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ is captured by a $k^{th}$ order symmetric tensor with the entry with index $i_1i_2...i_k$ being given by $\frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_k}}(\mathbf{x}_0)$
  - $x_j$ is the $j^{th}$ of the scalar variables in $\mathbf{x}$

- A $k^{th}$ order tensor $\in \mathbb{R}^n$ is an $n \times n \times \ldots \times n$ ($k$ times) array
  - A zero order tensor is a 1-entry ($1 \times 1$) array
  - A first order tensor is an $n \times 1$ array (column vector)
  - A second order tensor is an $n \times n$ array (matrix)
  - A third order tensor is an $n \times n \times n$ array
  - Etc.

- Pre- (post-) multiplying a $k^{th}$ order tensor by a row (column) vector yields a $(k-1)^{th}$ order tensor obtained by taking the dot-product between that vector and each of the tensor columns with first (last) index
  - For example if $k=1$, the result is a 1-entry ($0^{th}$ order) tensor obtained by taking the dot product of the vector and the column vector forming the 1-tensor
  - For example if $k=2$, the result of a pre-multiplication by a vector is a 1-tensor (vector) with its $i^{th}$ element obtained by taking the dot product of the vector and the $i^{th}$ column vector of the 2-tensor (matrix); and the result of a post-multiplication by a vector is a 1-tensor (vector) with its $i^{th}$ element obtained by taking the dot product of the vector and the $i^{th}$ row vector of the 2-tensor (matrix)
Taylor Series and Spatial Derivatives

- Taylor series express an image with local accuracy as a polynomial whose coefficients are derivatives normalized by factorials
  - In 1D, \( I(x) \approx I(x_0) + \sum_{k=0}^{n} \frac{d^k I/dx^k(x_0)}{k!} (x-x_0)^k \)
- For an image of arbitrary dimension \( M \), e.g., 2D (\( M=2 \)) or 3D (\( M=3 \)), all \( k \)th derivative values at a point \( x_0 \) are captured by the \( k \)-tensor \( D^k I(x_0) \), a linear operator represented by an array of size \( M \times M \times \ldots \times M \) (with dimension \( k \)) with the array element \( i_1, i_2, \ldots, i_k \) being \( \partial^k I/\partial x_{i_1}\partial x_{i_2} \ldots \partial x_{i_k}(x_0) \)
  - Examples:
    - for \( k=1 \), \( D^1 I(x_0) \) is the gradient vector with entries \( \partial I/\partial x_i \)
    - for \( k=2 \), \( D^2 I(x_0) \) is the Hessian matrix with entries \( \partial^2 I/\partial x_{i_1}\partial x_{i_2} \)
- The \( k \)th directional derivative at \( x_0 \) w.r.t. to the directions \( U = u_{i_1} u_{i_2} \ldots u_{i_k} \) is a scalar obtained by the successive linear operations of vector-product of \( D^k I(x_0) \) by \( u_{i_j} \) by the \( k \)-tensor \( D^k I(x_0) \), \( j = 1, 2, \ldots, k \) (or in any other order)
  - Call this operation \( U \bullet D^k I(x_0) \)
  - The order of the operations does not matter; all the operations but the last are on the left (mult. by \( u^T \)), but for \( k>1 \) the last must be on the right (mult. by \( u \))
Taylor Series and Spatial Derivatives

- Taylor series express an image **locally** as a polynomial whose coefficients are derivatives normalized by factorials
  - In 1D $I(x) \approx I(x_0) + \sum_{k=1}^{n} \left[ \frac{d^k I}{dx^k}(x_0) / k! \right] (x-x_0)^k$

- For an image of arbitrary dimension $M$, e.g., 2D ($M=2$) or 3D ($M=3$), the Taylor series approximation for $I(x)$ with $x$ near $x_0$ is
  
  $$I(x) \approx I(x_0) + \sum_{k=1}^{n} \sum_{\text{all } M^k \text{ choices of } i_1, i_2, \ldots, i_k} \left[ U_j \bullet D^k I(x_0) / k! \right] \times \prod_{j=1}^{k} (x_{i_j} - x_{0_{i_j}}),$$

  where each $i_m$ is chosen from 1, 2, ..., $M$, $\sum_{j=1}^{n} i_j = k$, and $U_j = u_{i_1} u_{i_2} \ldots u_{i_k}$
Taylor Series and Spatial Derivatives

- Taylor series express an image **locally** as a polynomial whose coefficients are derivatives normalized by factorials
  - Alternative form to
    
    \[
    I(x) \approx I(x_0) + \sum_{k=0}^{n} \frac{1}{k!} \sum_{\text{all } M_k \text{ choices of } i_1, i_2, \ldots, i_k} \left[ U_j \bullet D^k I(x_0) \right] \times \prod_{j=1}^{k} (x_{ij} - x_{0ij}),
    \]
    where each \( i_m \) is chosen from 1, 2, \ldots, \( k \)
    and \( U_j = u_{i_1} u_{i_2} \ldots u_{i_k} \) is as follows:
  - [This is the form we actually use.] Consider the vector \((x-x_0)\).
    The Taylor series approximation for \( I(x) \) with \( x \) near \( x_0 \) is
    
    \[
    I(x) \approx I(x_0) + \sum_{k=0}^{n} \text{term}_k / k!,
    \]
    where
    
    \[
    \text{term}_k = (x-x_0)^T \bullet \left[ (x-x_0)^T \bullet \ldots \left[ (x-x_0)^T \bullet D^k I(x_0) \right] \ldots \right] \bullet (x-x_0),
    \]
    with the series of vector products including \( k-1 \) entries
  - \( \prod_{j=1}^{k} (x_{ij} - x_{0ij}) \) are **basis images**; \( U_j \bullet D^k I(x_0) / k! \) are coefficients
Representation of 2D and 3D images as Taylor series

• Thus if we can compute image derivatives,

\[
I(x) = I(x_0) + \frac{1}{1!} \sum_{i=1}^{M} (x_i - x_{0i}) \frac{\partial I}{\partial x_i}(x_0) + \frac{1}{2!} \sum_{i,j=1}^{M} (x_i - x_{0i})(x_j - x_{0j}) \frac{\partial^2 I}{\partial x_i \partial x_j}(x_0) + \ldots
\]

\[
I(x) = I(x_0) + \frac{1}{1!} (x - x_0)^T D^1 I(x_0) + \frac{1}{2!} (x - x_0)^T D^2 I(x_0) (x - x_0) + \ldots
\]

– where \( D^1 I = \nabla I \), the gradient of I, an M-vector with components \( \frac{\partial I}{\partial x_i} \)

\[
D^2 I(x_0) = \text{the “Hessian” of I, an } M \times M \text{ matrix with components } \frac{\partial^2 I}{\partial x_i \partial x_j}
\]

– Each image derivative is itself an image

• When you cut off the series to a given number of terms, the result is a continuous image with pixel values computable at any value of \( x \)
Usefulness of Taylor Series and Spatial Derivatives

- Taylor series express an image **locally** as a polynomial in the components of $x$ whose coefficients are derivatives normalized by factorials
  - They provide a very poor approximation with a reasonably small number of terms except extremely locally
- The spatial derivatives of order 1 are used to compute edgeness, and those of order 2 are used to compute barness
Exercise
Due as a digital picture at class time on Thurs. 1/20

• Example: assume you are given the 2D image of 2-tensors $D^2 I(x)$
  - Reminder: $D^2 I(x)_{ij} = \frac{\partial^2 I}{\partial x_i \partial x_j}(x), i=1,2; j=1,2$

  Given two unit (direction) vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ and $D^2 I(x)$, a) give a formula for the (scalar) image $D^2 I_{u_1 u_2}(x)$
  Answer: $\mathbf{u}_1^T D^2 I(x) \mathbf{u}_2$

  b) give a formula for $\frac{\partial^2 I}{\partial y^2}(x)$ Answer: $\mathbf{e}_y^T D^2 I(x) \mathbf{e}_y$, where $\mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;
  thus $D^2 I(x)_{22}$

• Exercise: Given three unit (direction) vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ and the derivative tensor needed,
  a) give a formula for the (scalar) image $D^3 I_{u_1 u_2 u_3}(x)$

  b) ) give a formula for the scalar image $\frac{\partial^3 I}{\partial x^2 \partial y}(x)$