pp. 178, Exercise Set 4.3

38. *c.* $m = 3 \cdot 7 \cdot 11$

product $= 2^2 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot m = 2^2 \cdot 3^6 \cdot 7^2 \cdot 11^2 = (2 \cdot 3^3 \cdot 7 \cdot 11)^2 = 4158^2$

41. Note that $45^8 \cdot 88^5 = (3^2 \cdot 5)^8 \cdot (2^3 \cdot 11)^5 = (3^{16} \cdot 5^8) \cdot (2^{15} \cdot 11^5) = 2^{15} \cdot 3^{16} \cdot 5^8 \cdot 11^5$.

When this number is written in ordinary decimal form, each 0 at its end comes from a factor of 10, or one factor of 2 and one factor of 5.

Since there are at least eight factors of 2 but only eight factors of 5, there are exactly eight factors of 10 in the number.

This implies that the number ends with 8 zeroes.

47. <u>Proof</u>:

Suppose n is any nonnegative integer for which the sum of the digits of n is divisible by 9.

By definition of decimal representation, n can be written in the form

$$n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_2 10^2 + d_1 10 + d_0$$

where k is a nonnegative integer and all the d_i are integers from 0 to 9 inclusive. Then

$$\begin{aligned}
n &= d_k(\underbrace{99\ldots9}_{k\text{ 9's}}+1) + d_{k-1}(\underbrace{99\ldots9}_{(k-1)\text{ 9's}}+1) + \cdots + d_2(99+1) + d_1(9+1) + d_0 \\
&= d_k \cdot \underbrace{99\ldots9}_{k\text{ 9's}} + d_{k-1} \cdot \underbrace{99\ldots9}_{(k-1)\text{ 9's}} + \cdots + d_2 \cdot 99 + d_1 \cdot 9 + (d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0) \\
&= d_k \cdot \underbrace{11\ldots1}_{k\text{ 1's}} \cdot 9 + d_{k-1} \cdot \underbrace{11\ldots1}_{(k-1)\text{ 1's}} \cdot 9 + \cdots + d_2 \cdot 11 \cdot 9 + d_1 \cdot 9 + (d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0) \\
&= 9(d_k \cdot \underbrace{11\ldots1}_{k\text{ 1's}} + d_{k-1} \cdot \underbrace{11\ldots1}_{(k-1)\text{ 1's}} + \cdots + d_2 \cdot 11 + d_1) + (d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0) \\
&= (\text{an integer divisible by 9}) + (\text{the sum of the digits of } n).
\end{aligned}$$

Since the sum of the digits of n is divisible by 9, n can be written as a sum of two integers each of which is divisible by 9.

It follows from exercise 15 that n is divisible by 9.

pp. 189, Exercise Set 4.4

29. *a.* <u>Proof</u>: Suppose n is any integer. By the quotient-remainder theorem with $d = 3$, we know that $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$ for some integer q.

Case 1 ($n = 3q$ for some integer q): In this case,

$$\begin{aligned}
n^2 &= (3q)^2 && \text{by substitution} \\
&= 3(3q^2) && \text{by algebra.}
\end{aligned}$$

Let $k = 3q^2$. Then k is an integer because it is a product of integers. Hence $n^2 = 3k$ for some integer k.

Case 2 (n = 3q + 1 for some integer q): In this case,

$$\begin{aligned} n^2 &= (3q+1)^2 && \text{by substitution} \\ &= 9q^2 + 6q + 1 \\ &= 3(3q^2 + 2q) + 1 && \text{by algebra.} \end{aligned}$$

$n^2 = (3q+1)^2 = 9q^2+6q+1 = 3(3q^2+2q)+1$. Let $k = 3q^2+2q$. Then k is an integer because sums and products of integers are integers. Hence $n^2 = 3k + 1$ for some integer k.

Case 3 (n = 3q + 2 for some integer q): In this case,

$$\begin{aligned} n^2 &= (3q+2)^2 && \text{by substitution} \\ &= 9q^2 + 12q + 4 \\ &= 9q^2 + 12q + 3 + 1 \\ &= 3(3q^2 + 4q + 1) + 1 && \text{by algebra.} \end{aligned}$$

Let $k = 3q^2 + 4q + 1$. Then k is an integer because sums and products of integers are integers. Hence $n^2 = 3k + 1$ for some integer k.

Conclusion: In all three cases, either $n^2 = 3k$ or $n^2 = 3k + 1$ for some integer k *[as was to be shown]*.

b. Given any integer *n*, n^2 *mod* 3 = 0 or n^2 *mod* 3 = 1.

(Or, equivalently, given any integer *n*, n^2 *mod* 3 ≠ 2.)