pp. 129, Exercise Set 3.3

21. *c.* Statement (1) is true because $x^2 - 2xy + y^2 = (x-y)^2$. Thus given any real number x, take $y = x$, then $x - y = 0$, and so $x^2 - 2xy + y^2 = 0$.

Statement (2) is false. Given any real number x, choose a real number y with $y \neq x$. Then $x^2 - 2xy + y^2 = (x-y)^2 \neq 0$.

d. Statement (1) is true because no matter what real number x might be chosen, y can be taken to be 1 so that $(x-5)(y-1) = (x-5) \cdot 0 = 0$.

Statement (2) is also true. Take $x = 5$. Then for all real numbers y, $(x-5)(y-1) = 0(y-1) = 0$.

e. Statements (1) and (2) are both false because all real numbers have nonnegative squares and the sum of any two nonnegative real numbers is nonnegative. Hence for all real numbers x and y, $x^2 + y^2 \neq -1$.

56. These statements do not necessarily have the same truth values. For instance, let $D = \mathbf{R}$, the set of all real numbers, let $P(x)$ be "x is positive," and let $Q(x)$ be "x is negative." Then "$\exists x \in D, (P(x) \wedge Q(x)$" can be written "$\exists$ a real number x such that x is both positive and negative," which is false. On the other hand, "$(\exists x \in D,\ P(x)) \wedge (\exists x \in D, Q(x))$ can be written "$\exists$ a real number that is positive and $\exists$ a real number that is negative," which is true.

57. These statements do not necessarily have the same truth values. For example, let $D = \mathbf{Z}$, the set of all integers, let $P(x)$ be "x is even," and let $Q(x)$ be "x is odd." Then the statement "$\forall x \in D, (P(x) \vee Q(x))$" can be written "$\forall$ integers x, x is even or x is odd," which is true. On the other hand, "$(\forall x \in D, P(x)) \vee (\forall x \in D,\ Q(x))$" can be written "All integers are even or all integers are odd," which is false.

58. These statements have the same truth values for all domains D and predicates $P(x)$ and $Q(x)$.

If the statement "$\exists x \in D, (P(x) \vee Q(x))$" is true, then by definition of the truth values for $\exists$, the predicate $P(x) \vee Q(x)$ is true for at least one element x in D. Let's call such an element x_0. Then $P(x_0) \vee Q(x_0)$ is true, and so by definition of the truth values for $\vee$, at least one of $P(x_0)$ or $Q(x_0)$ is true. In case $P(x_0)$ is true, then the statement "$\exists x \in D,\ P(x)$" is true. In case $Q(x_0)$ is true, then the statement "$\exists x \in D,\ Q(x)$" is true. Since at least one of these cases must occur, the statement "$(\exists x \in D,\ P(x)) \vee (\exists x \in D,\ Q(x))$" is true by definition of truth values for $\vee$.

If the statement "$(\exists x \in D,\ P(x)) \vee (\exists x \in D,\ Q(x))$" is true, then by definition of truth values for $\vee$, at least one of the statements "$\exists x \in D,\ P(x)$" or "$\exists x \in D,\ Q(x)$" is true. In case "$\exists x \in D,\ P(x)$" is true, then by definition of truth values for $\exists$, there exists an element, say x_1, in D such that $P(x_1)$ is true. Then by definition of the truth values for $\vee$, $P(x_1) \vee Q(x_1)$ is true, and so by definition of the truth values for $\exists$, "$\exists x, (P(x) \vee Q(x))$" is true. Similarly, in case "$\exists x \in D,\ Q(x)$" is true, then by definition of truth values for $\exists$, there exists an element, say x_2, in D such that $Q(x_2)$ is true. It follows by definition of the truth values for $\vee$ that $P(x_2) \vee Q(x_2)$ is true, and so by definition of the truth values for $\exists$, "$\exists x, (P(x) \vee Q(x))$" is true. Since one of the two cases must occur, we can conclude that the statement "$\exists x \in D, (P(x) \vee Q(x)$" is true.