Quiz 2

(80') Name______________________________ PID______________________________

(16') 1. Determine whether each of the following statement is true or false. If it is false, please provide a counterexample. (Note that, due to time constraints, you do not need to prove a true statement; however, you must give a counterexample for false statement. Claiming a statement is false without a counterexample will receive no point.)

(a) The sum of two irrational numbers is irrational.
False.
Counterexample: $(\sqrt{2}) + (-\sqrt{2}) = 0$

(b) The product of two irrational number2 is irrational.
False.
Counterexample: $(\sqrt{2}) \cdot (-\sqrt{2}) = 2$

(c) The sum of a rational number and an irrational number is irrational.
True.
Proof: We prove this statement by contradiction. Suppose the statement is false. That is, suppose there is a rational number x and an irrational number y such that their sum, $x+y$, is rational. By the definition of rational numbers, we have

$x = a/b$ and $x+y = c/d$ for some integers $a,b,c,d,$ where $b\neq 0$ and $d\neq 0$.

That is,

$a/b+y= c/d$.

Then, by basic algebra,

$$y = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}.$$

Because the set of all integers are closed under addition, subtraction, and multiplication, and a,b,c,d are integers, $bc\text{-}ad$ and bd are also integers. Furthermore, because $b\neq 0$ and $d\neq 0$, $bd\neq 0$.
Therefore, by the definition of rational numbers, y is a rational number.
On the other hand, in the supposition at the beginning, we supposed y is an irrational number.
Thus, y is a rational number and y is an irrational number, which is a contradiction.
So, the supposition cannot be true. That is, the original statement is true.

(d) The product of a rational number and an irrational number is irrational.
False.
Counterexample: $0 \cdot (\sqrt{2}) = 0$

(4') 2. Prove the following statement by contradiction: $\sqrt{2}+\sqrt{3}$ is irrational. You can assume that "$\sqrt{2}$ is irrational" and "$\sqrt{2}+\sqrt{3}>0$" are already proved fact (i.e., you can use them in your proof).

Proof: We prove this statement by contradiction. Suppose the statement is false. That is, suppose $\sqrt{2}+\sqrt{3}$ is rational. Then, by the definition of rational numbers, we have

$$\sqrt{2}+\sqrt{3}=\frac{a}{b},$$

where a,b are integers and $a\neq 0$ (because $\sqrt{2}+\sqrt{3}>0$) and $b\neq 0$.

By multiplying b on both sides, we have

$$b\cdot\sqrt{2}+b\cdot\sqrt{3}=a,$$

which can be rearranged as

$$b\cdot\sqrt{3}=a-b\cdot\sqrt{2}.$$

Taking the square of both sides, we have

$$3b^2=a^2-2\sqrt{2}\cdot ab+2b^2,$$

which can be rearranged as

$$\sqrt{2}=\frac{a^2-b^2}{2ab}.$$

Because a,b are integers and $a\neq 0$ and $b\neq 0$, a^2-b^2 and $2ab$ are also integers, and $2ab\neq 0$. Therefore, by the definition of rational numbers, $\sqrt{2}$ is rational. However, it has been proved that $\sqrt{2}$ is irrational.

Thus, $\sqrt{2}$ is rational and $\sqrt{2}$ is irrational, which is a contradiction.

So, the supposition cannot be true. That is, the original statement is true.

(Bonus 5') Any comments and/or suggestions to this course and/or the instructor? (E.g. were the homework questions too hard or too simple?) You may use the other side of this paper.