

Midterm Exam Solution

Name _____ PID _____

Honor Code Pledge: I certify that I am aware of the Honor Code in effect in this course and observed the Honor Code in the completion of this exam.

Signature _____

Notes:

1. This exam is closed-book. Any collaboration or discussion during the exam is considered as a serious violation of the Honor Code.
2. Using any electronic devices is strictly forbidden during the exam, and is also considered as a serious violation of the Honor Code.
3. You are expected to write your solution neatly. Illegible solutions will not be graded.
4. In the proofs, you can assume the following are facts that you can directly use.
 - (a) Basic Algebra.
 - (b) The sum, difference, and product of any two integers are still an integer.
 - (c) Logical Equivalences listed in Theorem 2.1.1. (You are expected to know them without referring to the textbook.)
 - (d) An integer is either even or odd.
 - (e) The Quotient-Remainder Theorem as follows.

Theorem 4.4.1 The Quotient-Remainder Theorem

Given any integer n and positive integer d , there exist unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \leq r < d.$$

- (f) All definitions we learned, such as “rational,” “divide.”
- (g) Any other things specified in each particular question.

(16') 1. Write a negation for each of the following statement.

(a) Today is raining but we have a midterm today.

Today is not raining or we do not have a midterm today.

(b) UNC wins the championship only if UNC beats Duke.

The original statement is equivalent to "If UNC does not beat Duke, then UNC does not win the championship."

So, the negation is "UNC does not beat Duke but (and) UNC wins the championship"

(c) $\forall x \in \mathbf{Z}, -3 < x \leq 7$

$\exists x \in \mathbf{Z}, x \leq -3 \vee x > 7$

(d) $\forall m \in \mathbf{Z}, \exists n \in \mathbf{Z}$ such that if $m = n^2$, then $n = -m$.

$\exists m \in \mathbf{Z}$ such that $\forall n \in \mathbf{Z}, m = n^2$ and $n \neq -m$.

(10') 2. Determine whether the following argument form is valid or invalid by constructing a truth table. Please include a few words explaining how the truth table supports your conclusion.

$$\begin{aligned}
 & p \leftrightarrow q \\
 & p \rightarrow \sim r \\
 & r \rightarrow \sim q \\
 \therefore & q \vee r
 \end{aligned}$$

p	q	r	$\sim q$	$\sim r$	$p \leftrightarrow q$	$p \rightarrow \sim r$	$r \rightarrow \sim q$	$q \vee r$
T	T	T	F	F	T	F	F	T
T	T	F	F	T	T	T	T	T
T	F	T	T	F	F	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	F	F	T	F	T
F	T	F	F	T	F	T	T	T
F	F	T	T	F	T	T	T	T
F	F	F	T	T	T	T	T	F

The argument form is invalid, because in one of the three critical rows (bold font), the conclusion is false.

(6') 3. Write the converse, inverse, and contrapositive of "If Jim is my advisor, then I can finish my dissertation and I can enjoy my life at UNC."

Converse: If I can finish my dissertation and I can enjoy my life at UNC, then Jim is my advisor.

Inverse: If Jim is not my advisor, then I cannot finish my dissertation or I cannot enjoy my life at UNC.

Contrapositive: If I cannot finish my dissertation or I cannot enjoy my life at UNC, then Jim is not my advisor.

(10') 4. Construct a truth table for $p \leftrightarrow (q \rightarrow r)$ and write its corresponding disjunctive normal form (DNF) and conjunctive normal form (CNF) expressions.

p	q	r	$q \rightarrow r$	$p \leftrightarrow (q \rightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	F

DNF: $(p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r)$

CNF: $(\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

(9') 5. The NOR logic operator is denoted by the symbol \downarrow (called Peirce arrow). Its truth table is as follows, and its logical equivalence is $p \downarrow q \equiv \sim(p \vee q)$.

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Use the operator \downarrow only to rewrite logical equivalent expressions for $\sim p$, $p \wedge q$, and $p \vee q$. That is, $\sim p \equiv ?$, $p \wedge q \equiv ?$, and $p \vee q \equiv ?$ where \downarrow is the only allowed operator on the right-hand side of these " \equiv "; nonetheless, parentheses are allowed.

(Hint: $p \vee p \equiv p$, $\sim(\sim p) \equiv p$, $\sim(\sim p \vee \sim q) \equiv p \wedge q$).

$\sim p \equiv p \downarrow p$

$p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$, because $(p \downarrow p) \downarrow (q \downarrow q) \equiv (\sim p) \downarrow (\sim q) \equiv \sim(\sim p \vee \sim q) \equiv p \wedge q$

$p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$, because $(p \downarrow q) \downarrow (p \downarrow q) \equiv \sim(p \downarrow q) \equiv \sim(\sim(p \vee q)) \equiv p \vee q$

(20') 6. Determine whether each of the following statements is true or false. If it is true, prove it; if it is false, disprove it by a counterexample.

(a) For any integer n , if 3 does not divide n , then 3 divides $n^2 - 1$.

True.

Proof: Let n be a particular but arbitrarily chosen integer. Then, by the Quotient-Remainder Theorem, $n=3k$ for some integer k , or $n=3k+1$ for some integer k , or $n=3k+2$ for some integer k . Because 3 does not divide n , $n=3k$ for some integer k cannot be the case. We discuss the remaining two cases.

Case 1: $n=3k+1$ for some integer k . In this case,

$$n^2 - 1 = (3k + 1)^2 - 1 = 9k^2 + 6k + 1 - 1 = 3(3k^2 + 2k)$$

Because k is an integer, $3k^2 + 2k$ is also an integer. Therefore, 3 divides $n^2 - 1$ in this case.

Case 2: $n=3k+2$ for some integer k . In this case,

$$n^2 - 1 = (3k + 2)^2 - 1 = 9k^2 + 12k + 4 - 1 = 3(3k^2 + 4k + 1)$$

Because k is an integer, $3k^2 + 4k + 1$ is also an integer. Therefore, 3 divides $n^2 - 1$ in this case.

Combining Cases 1 and 2, we can conclude that, if 3 does not divide n , then 3 divides $n^2 - 1$.

(b) The sum of two irrational numbers is zero or irrational.

False.

Counterexample: $(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4$, where $(2 + \sqrt{2})$ and $(2 - \sqrt{2})$ are both irrational numbers, and their sum is 4, which is not zero and is not irrational.

(c) The product of a **non-zero** rational number and an irrational number is irrational.

True.

Proof: we prove this statement by contradiction. Suppose the statement is false. That is, suppose there are a non-zero rational number x and an irrational number y such that their product xy is rational. Then, by the definition of rational numbers,

$$x = \frac{a}{b} \text{ and } xy = \frac{c}{d} \text{ for some integers } a, b, c, d \text{ where } b \neq 0 \text{ and } d \neq 0.$$

That is, $\frac{a}{b} \cdot y = \frac{c}{d}$. Because x is non-zero, $a \neq 0$.

Therefore, $y = \frac{c}{a} \cdot \frac{b}{a} = \frac{bc}{ad}$. Because a, b, c, d are integers, bc and ad are both integers. Also, because $a \neq 0$ and $d \neq 0$, $ad \neq 0$.

So, by the definition of rational numbers, y is a rational number, which contradicts the supposition that y is an irrational number.

Thus, the supposition cannot be true, and the original statement is true.

(10*) 7. Use mathematical induction to prove the following generalized De Morgan's Law for arbitrary number of statement variables. That is, prove that for any integer $n \geq 2$,

$$\sim(p_1 \wedge p_2 \wedge \dots \wedge p_n) \equiv \sim p_1 \vee \sim p_2 \vee \dots \vee \sim p_n.$$

You can assume the two-variable De Morgan's Law, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, is an already proven fact. You can also assume the commutative law and the associative law are also already proven for multiple " \wedge " or multiple " \vee ". (In case you don't remember the laws' names: commutative law—the order of variables doesn't matter; associative law—adding parentheses doesn't matter.)

Proof: we prove it by mathematical induction. Let $P(n)$ denote that

$$\sim(p_1 \wedge p_2 \wedge \dots \wedge p_n) \equiv \sim p_1 \vee \sim p_2 \vee \dots \vee \sim p_n.$$

Basis Step: we consider $P(2)$, which is $\sim(p_1 \wedge p_2) \equiv \sim p_1 \vee \sim p_2$. By the two-variable De Morgan's Law, it is true. So, $P(2)$ is true.

Inductive Step: we suppose $P(k)$ is true for some particular integer $k \geq 2$. That is,

$$\sim(p_1 \wedge p_2 \wedge \dots \wedge p_k) \equiv \sim p_1 \vee \sim p_2 \vee \dots \vee \sim p_k.$$

[Above is our Inductive Hypothesis.]

[We must show that $P(k+1)$ is also true. That is ,

$$\sim(p_1 \wedge p_2 \wedge \dots \wedge p_k \wedge p_{k+1}) \equiv \sim p_1 \vee \sim p_2 \vee \dots \vee \sim p_k \vee \sim p_{k+1}.]$$

$$\begin{aligned} \sim(p_1 \wedge p_2 \wedge \dots \wedge p_k \wedge p_{k+1}) &\equiv \sim((p_1 \wedge p_2 \wedge \dots \wedge p_k) \wedge p_{k+1}) \\ &\equiv \sim(p_1 \wedge p_2 \wedge \dots \wedge p_k) \vee \sim p_{k+1} && \text{by the two-variable De Morgan's Law} \\ &\equiv (\sim p_1 \vee \sim p_2 \vee \dots \vee \sim p_k) \vee \sim p_{k+1} && \text{by the Inductive Hypothesis} \\ &\equiv \sim p_1 \vee \sim p_2 \vee \dots \vee \sim p_k \vee \sim p_{k+1} \end{aligned}$$

That is, $P(k+1)$ is true.

Thus, by induction, $P(n)$ is true for all integers $n \geq 2$.

(9*) 8. Calculate the following sum or product. (I encourage you to include intermediate steps, which may give you partial credits in case you did the math wrong.)

(a)

$$\sum_{i=3}^5 (i^2 - 5)$$

(b)

$$\prod_{k=0}^3 (k! + k)$$

(c)

$$\sum_{i=1}^3 \sum_{j=i}^4 (i \cdot j)$$

(a) $\sum_{i=3}^5 (i^2 - 5) = (9 - 5) + (16 - 5) + (25 - 5) = 4 + 11 + 20 = 35$

(b) $\prod_{k=0}^3 (k! + k) = (1 + 0) * (1 + 1) * (2 * 1 + 2) * (3 * 2 * 1 + 3) = 1 * 2 * 4 * 9 = 72$

(c) $\sum_{i=1}^3 \sum_{j=i}^4 (i \cdot j) = \sum_{j=1}^4 (1 \cdot j) + \sum_{j=2}^4 (2 \cdot j) + \sum_{j=3}^4 (3 \cdot j)$

$$= 1 * (1 + 2 + 3 + 4) + 2 * (2 + 3 + 4) + 3 * (3 + 4) = 10 + 18 + 21 = 49$$

(10') 9. A sequence a_0, a_1, a_2, \dots is defined recursively as follows:

$$a_0 = 2, a_1 = 9$$

$$a_k = 5a_{k-1} - 6a_{k-2} \text{ for all integers } k \geq 2.$$

Use strong mathematical induction to prove that for all integers $n \geq 0$, $a_n = 5 \cdot 3^n - 3 \cdot 2^n$.

Proof: we prove this by strong mathematical induction. Let $P(n)$ denote $a_n = 5 \cdot 3^n - 3 \cdot 2^n$.

Basis Step: we consider both $P(0)$ and $P(1)$.

The LHS of $P(0)$ is 2, and the RHS of $P(0)$ is $5 \cdot 3^0 - 3 \cdot 2^0 = 5 - 3 = 2$. So, $P(0)$ is true.

The LHS of $P(1)$ is 9, and the RHS of $P(1)$ is $5 \cdot 3^1 - 3 \cdot 2^1 = 15 - 6 = 9$. So, $P(1)$ is true.

Inductive Step: we suppose $P(0), P(1), \dots, P(k)$ are all true for integer $k \geq 1$. That is,

$$a_i = 5 \cdot 3^i - 3 \cdot 2^i \text{ for any integer } i \text{ such that } 0 \leq i \leq k.$$

[Above is our Inductive Hypothesis.]

[We want to show that $P(k+1)$ is also true. That is, $a_{k+1} = 5 \cdot 3^{k+1} - 3 \cdot 2^{k+1}$.]

$$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} && \text{by the recursive definition} \\ &= 5 \cdot (5 \cdot 3^k - 3 \cdot 2^k) - 6 \cdot (5 \cdot 3^{k-1} - 3 \cdot 2^{k-1}) && \text{by the Inductive Hypothesis} \\ &= 75 \cdot 3^{k-1} - 30 \cdot 2^{k-1} - 30 \cdot 3^{k-1} + 18 \cdot 2^{k-1} && \text{by basic algebra} \\ &= 45 \cdot 3^{k-1} - 12 \cdot 2^{k-1} \\ &= 5 \cdot 9 \cdot 3^{k-1} - 3 \cdot 4 \cdot 2^{k-1} \\ &= 5 \cdot 3^{k+1} - 3 \cdot 2^{k+1} \end{aligned}$$

That is, $P(k+1)$ is true.

Thus, by induction, $P(n)$ is true for all integers $n \geq 0$.