Homework 4

Due on Monday, 6/5, 1:15 PM in class

Name________________________________ PID________________________________

Honor Code Pledge: I certify that I am aware of the Honor Code in effect in this course and observed the Honor Code in the completion of this homework.

Signature____________________________

(30') 1. Calculate the following sum or product. (I encourage you to include intermediate steps, which may give you partial credits in case you did the math wrong.)

(a)
$$\sum_{i=3}^{7}(i-5)$$

(b)
$$\prod_{k=0}^{4}(k!)$$

(c)
$$\left(\sum_{j=6}^{6} j\right)^2$$

(d)
$$\sum_{i=2}^{5}\sum_{j=1}^{3}(i \cdot j)$$

(e)
$$\prod_{i=1}^{3}\left(\sum_{j=1}^{i} j\right)$$

(f)
$$\sum_{i=2}^{5}\sum_{j=1}^{i}(i \cdot j)$$

Solution:

(a) $\sum_{i=3}^{7}(i-5)$ = (3-5)+(4-5)+(5-5)+(6-5)+(7-5) = 0

(b) $\prod_{k=0}^{4}(k!)$ = 0! * 1! * 2! *3! * 4! = 1 * 1 * (2*1) * (3*2*1)* (4*3*2*1) = 288

(c) $\left(\sum_{j=6}^{6} j\right)^2$ = 6^2 = 36

(d) $\sum_{i=2}^{5}\sum_{j=1}^{3}(i \cdot j) = \sum_{j=1}^{3}(2 \cdot j) + \sum_{j=1}^{3}(3 \cdot j) + \sum_{j=1}^{3}(4 \cdot j) + \sum_{j=1}^{3}(5 \cdot j)$

= (2*1+2*2+2*3) + (3*1+3*2+3*3) + (4*1+4*2+4*3) + (5*1+5*2+5*3)
= 12 + 18 + 24 + 30 = 84

(e) $\prod_{i=1}^{3}\left(\sum_{j=1}^{i} j\right) = \left(\sum_{j=1}^{1} j\right) \cdot \left(\sum_{j=1}^{2} j\right) \cdot \left(\sum_{j=1}^{3} j\right)$

$= (1) \cdot (1+2) \cdot (1+2+3)$ = 18

(f) $\sum_{i=2}^{5}\sum_{j=1}^{i}(i \cdot j) = \sum_{j=1}^{2}(2 \cdot j) + \sum_{j=1}^{3}(3 \cdot j) + \sum_{j=2}^{4}(4 \cdot j) + \sum_{j=2}^{5}(5 \cdot j)$

= (2*1+2*2) + (3*1+3*2+3*3) + (4*1+4*2+4*3+4*4)+ (5*1+5*2+5*3+ 5*4+5*5)
= 6 + 18 + 40 + 75 = 139

(20') 2. Consider the following formula where *n* is an integer and $n \geq 3$:

$$\sum_{i=3}^{n} i = \frac{(n-2)(n+3)}{2}$$

(a) Expand the Left-Hand-Side of the formula. (That is, rewrite it without the "Σ" but with "…")

(b) Prove the formula by mathematical induction.

Solution:

(a) 3+4+5+…+n

(b) *Proof*: we prove this formula by induction. Let P(n) denote the formula

$$\sum_{i=3}^{n} i = \frac{(n-2)(n+3)}{2}.$$

Basis Step: we consider P(3). The LHS of P(3) is 3, and the RHS of P(3) is $\frac{(3-2)(3+3)}{2} = 3$.

Therefore, P(3) is true.

Inductive Step: we assume P(k) is true for integer k≥3. That is,

$$\sum_{i=3}^{k} i = \frac{(k-2)(k+3)}{2}$$

[The above is our Inductive Hypothesis (IH).]

[We want to show that P(k+1) is also true. That is, $\sum_{i=3}^{k+1} i = \frac{(k+1-2)(k+1+3)}{2}$. Or, equivalently,

$\sum_{i=3}^{k+1} i = \frac{(k-1)(k+4)}{2}$. Also, note that $(k-1)(k+4) = k^2 + 3k - 4$.]

We now consider P(k+1).

$$\begin{aligned}\sum_{i=3}^{k+1} i &= \sum_{i=3}^{k} i + (k+1) \\ &= \frac{(k-2)(k+3)}{2} + (k+1) \qquad \{\text{by IH}\} \\ &= \frac{k^2+k-6}{2} + \frac{2k+2}{2} \\ &= \frac{k^2+3k-4}{2} \\ &= \frac{(k-1)(k+4)}{2}\end{aligned}$$

Therefore, P(k+1) is true.

Thus, combining the Basis Step and the Inductive Step, we can conclude that P(n) is true for all integers $n \geq 3$.

(15') 3. Prove the following statement by mathematical induction:

$7^n - 1$ is divisible by 6, for any integer $n \geq 0$.

Solution:

Proof: we prove this statement by induction. Let P(n) denote "$7^n - 1$ is divisible by 6."

Basis Step: we consider P(0). $7^0 - 1 = 1 - 1 = 0$, and 0 = 6*0. So, P(0) is true.

Inductive Step: we assume P(k) for integer k≥0. That is, $7^k - 1$ is divisible by 6.
Therefore, $7^k - 1 = 6 \cdot r$ for some integer r. That is, $7^k = 6 \cdot r + 1$ for some integer r.
[The above is our Inductive Hypothesis (IH).]
[We must show that P(k+1) is also true. That is, $7^{k+1} - 1$ is divisible by 6.]

$$\begin{aligned} 7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\ &= 7 \cdot (6 \cdot r + 1) - 1 \quad \text{\{by IH\}} \\ &= 42r + 6 \\ &= 6 \cdot (7r + 1) \end{aligned}$$

Because r is an integer, 7r+1 is also an integer. Therefore, $7^{k+1} - 1$ is divisible by 6.
That is, P(k+1) is true.
Thus, combining the Basis Step and the Inductive Step, we can conclude that P(n) is true for all integers $n \geq 0$.

(15') 4. Define a sequence $a_1, a_2, a_3, \ldots$ as: $a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$.
Use strong mathematical induction to prove that $a_n < \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Solution:

Proof: we prove this inequality by induction. Let P(n) denote $a_n < \left(\frac{7}{4}\right)^n$.

Basis Step: we consider P(1) and P(2). $a_1=1<\frac{7}{4} = \left(\frac{7}{4}\right)^1$, so P(1) is true. $a_2=3<\frac{49}{16} = \left(\frac{7}{4}\right)^2$, so P(2) is true. Thus, we have that both P(1) and P(2) are true.

Inductive Step: we assume P(1), P(2), …, P(k) are all true for integer k≥2.That is, $a_i < \left(\frac{7}{4}\right)^i$ is true for any integer i such that $1 \leq i \leq k$. [This is our Inductive Hypothesis (IH).]

[We must show that P(k+1) is also true. That is, $a_{k+1} < \left(\frac{7}{4}\right)^{k+1}$.]

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \quad \text{\{by the definition of this sequence\}} \\ &< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \quad \text{\{by IH\}} \\ &= \left(\frac{7}{4}\right) \cdot \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-1} \\ &= \left(\frac{11}{4}\right) \cdot \left(\frac{7}{4}\right)^{k-1} \\ &= \left(\frac{44}{16}\right) \cdot \left(\frac{7}{4}\right)^{k-1} \\ &< \left(\frac{49}{16}\right) \cdot \left(\frac{7}{4}\right)^{k-1} \\ &= \left(\frac{7}{4}\right)^2 \cdot \left(\frac{7}{4}\right)^{k-1} \end{aligned}$$

$$= \left(\frac{7}{4}\right)^{k+1}$$

Therefore, P(k+1) is true.

Thus, combining the Basis Step and the Inductive Step, we can conclude that P(n) is true for all integers $n \geq 1$.

(20') 5. Define a sequence $b_1, b_2, b_3, \ldots$ as: $b_1 = 2$, and $b_k = b_{k-1} + 2 \cdot 3^k$ for all integers $k \geq 2$.

(a) Calculate b_2, b_3, b_4.

(b) Use iteration to guess an explicit, closed-form formula. That is, express b_n as a function of n without "…", "Σ", or "Π".

(Hint: you might need to use the formula $1 + r + r^2 + \ldots + r^n = \frac{r^{n+1}-1}{r-1}$)

(c) Use to mathematical induction to prove the formula you derived in (b) above.

Solution:

(a) b_2=2+2*9=20, b_3=20+2*27=74, b_4=74+2*81=236

(b) We can try iterating b_n as follows, by the definition $b_k = b_{k-1} + 2 \cdot 3^k$ for all integers $k \geq 2$.

$$b_1 = 2$$
$$b_2 = b_1 + 2 \cdot 3^2 = 2 + 2 \cdot 3^2$$
$$b_3 = b_2 + 2 \cdot 3^3 = 2 + 2 \cdot 3^2 + 2 \cdot 3^3$$
$$b_4 = b_3 + 2 \cdot 3^4 = 2 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4$$

… …

$$b_n = 2 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots + 2 \cdot 3^n$$
$$= 2 + 2 \cdot 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{n-2}) = 2 + 18 \cdot \frac{3^{n-1} - 1}{3 - 1} = 2 + 9(3^{n-1} - 1)$$
$$= 3^{n+1} - 7$$

Therefore, we guess $b_n = 3^{n+1} - 7$ for all $n \geq 1$.

(c) *Proof*: we prove $b_n = 3^{n+1} - 7$ for all $n \geq 1$ by induction. Let P(n) denote $b_n = 3^{n+1} - 7$

Basis Step: we consider P(1). The LHS of P(1) is $b_1 = 2$, and the RHS of P(1) is $3^{1+1} - 7 = 9 - 7 = 2$. Therefore, P(1) is true.

Inductive Step: we assume P(k) is true for integer $k \geq 1$. That is, $b_k = 3^{k+1} - 7$.

[The above is our Inductive Hypothesis (IH).]

[We must show that P(k+1) is also true. That is, $b_{k+1} = 3^{k+2} - 7$]

$$b_{k+1} = b_k + 2 \cdot 3^{k+1} \quad \text{\{by the definition of the sequence\}}$$
$$= 3^{k+1} - 7 + 2 \cdot 3^{k+1} \quad \text{\{by IH\}}$$
$$= (1 + 2) \cdot 3^{k+1} - 7$$
$$= 3^{k+2} - 7$$

Therefore, P(k+1) is true.

Thus, combining the Basis Step and the Inductive Step, we can conclude that P(n) is true for all integers n ≥ 1.