Homework 3

Due on Friday, 6/2, 1:15 PM in class

Name_______________________________ PID_______________________________

Honor Code Pledge: I certify that I am aware of the Honor Code in effect in this course and observed the Honor Code in the completion of this homework.

Signature___________________________

Determine whether each of the following statements 1-4 is true or false. If it is true, prove it **from the definitions (nonetheless, the proof can be either direct or indirect)**; if it is false, disprove it by a counterexample.

(15') 1. ∀ integers m and n, if $2m + n$ is odd then m and n are both odd.

Solution:

False. *Counterexample*: Let m=2 and n=1. $2m+n = 5$, which is odd, but m=2 which is not odd.

(15') 2. For all integers n, $n^2 + n + 1$ is odd.

Solution:

True.

Proof: Let n be a particular but arbitrarily chosen integer. Then, n is either odd or even.

<u>Case 1</u>: n is odd.

In this case, by the definition of odd numbers, n=2k+1 for some integer k. Therefore,

$$n^2 + n + 1 = (2k+1)^2 + (2k+1) + 1 = 4k^2 + 6k + 3 = 2(2k^2 + 3k + 1) + 1.$$

Thus, by definition, $n^2 + n + 1$ is odd.

<u>Case 2</u>: n is even.

In this case, by the definition of odd numbers, n=2k for some integer k. Therefore,

$$n^2 + n + 1 = (2k)^2 + (2k) + 1 = 4k^2 + 2k + 1 = 2(2k^2 + k) + 1.$$

Thus, by definition, $n^2 + n + 1$ is odd.

Combining Case 1 and Case 2, we can conclude that, for all integers n, $n^2 + n + 1$ is odd.

(15') 3. For all real numbers r, if r^3 is irrational then r is irrational.

Solution:

True. (The following is a proof by contradiction, you can also do a proof by contraposition.)

Proof: we prove this statement by contradiction. Suppose the statement is false. That is, suppose there exists a real number r such that r^3 is irrational and r is rational.

Then, by the definition of rational numbers, $r = a/b$ for some integers a and b where $b \neq 0$.

By basic algebra, $r^3 = a^3/b^3$.

Because a and b are integers, a^3 and b^3 are also integers; because $b \neq 0$, $b^3 \neq 0$.

Therefore, by definition, r^3 is rational. This contradicts the supposition that r^3 is irrational. Thus, the supposition cannot be true, and the original statement is true

(15') 4. For all integers a and b, if $a \mid b^2$ and $a \leq b$, then $a \mid b$.

Solution:

False. *Counterexample*: Let a=4 and b=6. Because b^2=36 and $4 \mid 36$, $a \mid b^2$ and $a \leq b$ is true. However, 4 does not divide b, i.e, $a \nmid b$.

Prove the following statement. You can use the Quotient-Remainder Theorem. That is, assume Theorem 4.4.1 on pp. 180 in the textbook is already proven.

(20') 5. For all integer n, if $3 \mid n^2$ then $3 \mid n$.

(Hint: By contradiction and by division into cases while deriving the contradiction.)

Solution:

Proof: we prove this statement by contradiction. We suppose the statement is false. That is, we suppose there exists an integer n such that $3 \mid n^2$ and $3 \nmid n$. By the Quotient-Remainder Theorem, $n=3k$ for some integer k, or $n=3k+1$ for some integer k, or $n=3k+2$ for some integer k.

Because $3 \nmid n$, $n \neq 3k$ for any integer k. Therefore, $n=3k+1$ for some integer k, or $n=3k+2$ for some integer k.

<u>Case 1</u>: $n=3k+1$ for some integer k.

In this case, $n^2=(3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k) + 1$.

Therefore, $3 \nmid n^2$ (because by the Quotient-Remainder Theorem, it is impossible that both $n^2=3r$ for some integer r and $n^2=3s+1$ for some integer s).

<u>Case 2</u>: $n=3k+2$ for some integer k.

In this case, $n^2=(3k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1) + 1$.

Therefore, $3 \nmid n^2$

Combining Case 1 and Case 2, we can conclude that $3 \nmid n^2$. However, we supposed $3 \mid n^2$. Therefore, $3 \nmid n^2$ and $3 \mid n^2$, which is a contradiction. Thus, the supposition cannot be true, and the original statement is true.

Prove the following statement. You can use statement 5 above. That is, assume you have correctly proven the statement above.

(20') 6. $\sqrt{3}$ is irrational.

Solution:

Proof: we prove this statement by contradiction. We suppose the statement is false. That is, we suppose $\sqrt{3}$ is rational. By the definition of rational numbers, $\sqrt{3} = a/b$ for some integers a, b where a, b have no common factor (by dividing a and b by any common factors if necessary).

Squaring both sides of $\sqrt{3} = a/b$, we have $3 = a^2/b^2$, i.e., $3b^2 = a^2$. Therefore, $3 \mid a^2$.

By statement 5 above, we have $3 \mid a$. (*)

Therefore, $a = 3k$ for some integer k. Substituting in the equation $3b^2 = a^2$, we have $3b^2 = 9k^2$, which is $b^2 = 3k^2$. Therefore, $3 \mid b^2$.

By statement 5 above, we have $3 \mid b$. (**)

By (*) and (**), we have $3 \mid a$ and $3 \mid b$. That is, a and b have a common factor of 3. This contradicts the supposition that a and b have no common factor. Thus, the supposition cannot be true, and the original statement is true.