Final Exam Solution

Name________________________________ PID________________________________

Honor Code Pledge: I certify that I am aware of the Honor Code in effect in this course and observed the Honor Code in the completion of this exam.

Signature____________________________

Notes:

1. This exam is closed-book. Any collaboration or discussion during the exam is considered as a serious violation of the Honor Code.
2. Using any electronic devices except calculators is strictly forbidden during the exam, and is also considered as a serious violation of the Honor Code.
3. You are expected to write your solution neatly. Illegible solutions will not be graded.
4. In the proofs, you can assume the following are facts that you can directly use.
   (a) Basic Algebra.
   (b) The sum, difference, and product of any two integers are still an integer.
   (c) Logical Equivalences listed in Theorem 2.1.1. (You are expected to know them without referring to the textbook.)
   (d) An integer is either even or odd.
   (e) The Quotient-Remainder Theorem as follows.

> **Theorem 4.4.1 The Quotient-Remainder Theorem**
>
> Given any integer n and positive integer d, there exist unique integers q and r such that
>
> $$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

   (f) All definitions we learned, such as “rational,” “divide.”
   (g) Any other things specified in each particular question.

(6’) 1. Write a negation for each of the following statement.

(a) UNC does not beat Duke but UNC wins the championship.
Negation: UNC beats Duke or UNC does not win the championship.

(b) It is raining in Chapel Hill only if it is raining in Durham.
Negation: It is raining in Chapel Hill and it is not raining in Durham.

(c) $\exists n \in \mathbf{Z}$ such that $\forall m \in \mathbf{Z},\ -n < m < n$.
Negation: $\forall n \in \mathbf{Z},\ \exists m \in \mathbf{Z}$ such that $m \leq -n \vee m \geq n$.

(6’) 2. Write the converse, inverse, and contrapositive of “If today is a holiday, then no classes are held today.”
Converse: If no classes are held today, then today is a holiday.
Inverse: If today is not a holiday, then some classes are held today.
Contrapositive: If some classes are held today, then today is not a holiday.

(12’) 3. Determine whether each of the following statements is true or false. If it is true, prove it; if it is false, disprove it by a counterexample.
(a) The sum of three irrational numbers is irrational.
False.
Counterexample: $(\sqrt{2}) + (2 \cdot \sqrt{2}) + (-3 \cdot \sqrt{2}) = 0$.

(b) For any real number a, if a^2 is irrational, then a is irrational.
True.
Proof: we prove this statement by contradiction. Suppose the negation of the statement is true. That is, we suppose a^2 is irrational and a is rational.
Because a is rational, by the definition of rational numbers, $a = x/y$ for some integers x, y where $y \neq 0$. Then, $a^2 = x^2/y^2$. Because x, y are integers and $y \neq 0$, x^2, y^2 are also integers and $y^2 \neq 0$. Therefore, a^2 is rational by definition.
However, we supposed a^2 is irrational. That is, a^2 is both rational and irrational, which is a contradiction.
Thus, the supposition cannot be true and the original statement is true.

(c) For any integer n, $n^2 + 5$ is not divisible by 4.
True.
Proof: we consider two cases that n is either even or odd.

Case 1: n is even. In this case, $n = 2k$ for some integer k. So,

$$n^2 + 5 = (2k)^2 + 5 = 4k^2 + 5 = 4(k^2 + 1) + 1.$$

Because k is an integer, $k^2 + 1$ is also an integer. Therefore, by the Quotient-Remainder Theorem, $n^2 + 5$ cannot be re-written as $4m$ for some integer m. Thus, $n^2 + 5$ is not divisible by 4.

Case 2: n is odd. In this case, $n = 2k+1$ for some integer k. So,

$$n^2 + 5 = (2k + 1)^2 + 5 = 4k^2 + 4k + 1 + 5 = 4(k^2 + k + 1) + 2.$$

Because k is an integer, $k^2 + k + 1$ is also an integer. Therefore, by the Quotient-Remainder Theorem, $n^2 + 5$ cannot be re-written as $4m$ for some integer m. Thus, $n^2 + 5$ is not divisible by 4.

(7') 4. Use mathematical induction to prove that for any integer $n \geq 4,\ n! > 2^n$.
Proof: we prove this statement by induction. We let *P*(*n*) denote the inequality $2^n < n! < n^n$.

Basis Step: we consider P(4). $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,\ 2^4 = 16$. Because $24 > 16,\ 4! > 2^4$.
That is, P(4) is true.

Inductive Step: we suppose P(*k*) is true for some integer $k \geq 4$. That is, $k! > 2^k$.
[The above is our Inductive Hypothesis.]
[We need to show P(*k*+1) is also true, i.e., $(k+1)! > 2^{k+1}$.]
Because $k \geq 4$, $\mathrm{k} + 1 > 2$. Also, by the Inductive Hypothesis $k! > 2^k$.
So, $(k+1) \cdot k! > 2 \cdot 2^k$, i.e., $(k+1)! > 2^{k+1}$.
That is, P(*k*+1) is true.

Therefore, by induction, P(*n*) is true, for integer $n \geq 4$.

(7') 5. A sequence $a_0, a_1, a_2, \ldots$is defined recursively as follows:

$$a_0 = 5,\ a_1 = -5$$
$$a_k = 9a_{k-1} - 14a_{k-2} \text{ for all integers } k \geq 2.$$

Use strong mathematical induction to prove that for all integers $n \geq 0$, $a_n = 2^{n+3} - 3 \cdot 7^n$.
Proof: we prove this by strong mathematical induction. Let P(*n*) denote $a_n = 2^{n+3} - 3 \cdot 7^n$.

Basis Step: we consider both P(0) and P(1).
The LHS of P(0) is 5, and the RHS of P(0) is $2^{0+3} - 3 \cdot 7^0 = 8 - 3 = 5$. So, P(0) is true.
The LHS of P(1) is -5, and the RHS of P(0) is $2^{1+3} - 3 \cdot 7^1 = 16 - 21 = -5$. So, P(1) is true.

Inductive Step: we suppose P(0), P(1), …, P(*k*) are all true for integer $k \geq 1$. That is,

$$a_i = 2^{i+3} - 3 \cdot 7^i \text{ for any integer } i \text{ such that } 0 \leq i \leq k.$$

[Above is our Inductive Hypothesis.]
[We want to show that P(*k*+1) is also true. That is, $a_{k+1} = 2^{k+4} - 3 \cdot 7^{k+1}$.]

$$\begin{aligned}
a_{k+1} &= 9a_k - 14a_{k-1} \quad \text{by the recursive definition} \\
&= 9 \cdot \left(2^{k+3} - 3 \cdot 7^k\right) - 14 \cdot \left(2^{k-1+3} - 3 \cdot 7^{k-1}\right) \quad \text{by the Inductive Hypothesis} \\
&= 9 \cdot 2^{k+3} - 27 \cdot 7^k - 14 \cdot 2^{k+2} + 42 \cdot 7^{k-1} \quad \text{by basic algebra} \\
&= 4 \cdot 2^{k+2} - 147 \cdot 7^{k-1} \\
&= 2^2 \cdot 2^{k+2} - 3 \cdot 7^2 \cdot 7^{k-1} \\
&= 2^{k+4} - 3 \cdot 7^{k+1}
\end{aligned}$$

That is, P(*k*+1) is true.
Thus, by induction, P(*n*) is true for all integers $n \geq 0$.

(8') 6. Let A, B, C be three sets such that A= $\{a, b\}$, B= $\{a, b, c\}$, C=$\{a, b, \{a, b\}, \{a, b, c\}\}$. Also, we let Ø denote the empty set. Answer the following questions by "Yes" or "No."

(a) Is $A \in B$? **No**

(b) Is $A \subseteq B$? **Yes**

(c) Is $B \in C$? **Yes**

(d) Is $B \subseteq C$? **No**

(e) Is $A \in C$? **Yes**

(f) Is $A \subseteq C$? **Yes**

(g) Is $\emptyset \in C$? **No**

(h) Is $\emptyset \subseteq C$? **Yes**

(6') 7. Let set $S = \{\emptyset, \{x, y, z\}\}$. Write the power set of S and write $S \times S$.

Power set of S: $\{ \emptyset, \{\emptyset\}, \{\{x, y, z\}\}, \{\emptyset, \{x, y, z\}\} \}$.

$S \times S = \{ (\emptyset, \emptyset), (\emptyset, \{x, y, z\}), (\{x, y, z\}, \emptyset), (\{x, y, z\}, \{x, y, z\}) \}$

(12') 8. Define function f : $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$ as follows: $f(x, y) = (5y - 1, 7 - x)$

(a) $f(0, 0) = ?$ $f(1, 7) = ?$

(b) Is f injective? Prove or give a counterexample.

(c) Is f surjective? Prove or give a counterexample.

(d) Is f a bijection? If not, explain why not. If yes, find f^{-1}.

(e) Define another function $g : \mathbf{R} \rightarrow \mathbf{R}$ as follows: $g(x) = x^2$. Find $f(g(y), g(x)) = ?$

(a) $f(0, 0) = (-1, 7)$. $f(1, 7) = (34, 6)$.

(b) Yes. *Proof*: Suppose (x_1, y_1) and (x_2, y_2) are two elements in the domain such that $f(x_1, y_1) = f(x_2, y_2)$. By the definition of F, we know $5y_1 - 1 = 5y_2 - 1$ and $7 - x_1 = 7 - x_2$, from which, by basic algebra, we can conclude that $x_1 = x_2$ and $y_1 = y_2$. That is, $(x_1, y_1) = (x_2, y_2)$.

(c) Yes. *Proof*: Suppose (u, v) is a particular but arbitrarily chose element in the co-domain. Then, u and v are both real numbers. Let $x = 7 - v$ and $y = (u+1)/5$. Then, x and y are also both real numbers. So, (x, y) is in the domain. By the definition of F, we have

$$f(x,y) = f\left(7 - v, \frac{u+1}{5}\right) = \left(5 \cdot \frac{u+1}{5} - 1, 7 - (7 - v)\right) = (u, v).$$

(d) Yes, because f is both injective and surjective. By the steps in (c), we can find the inverse function

$$f^{-1}(u, v) = \left(7 - v, \frac{u+1}{5}\right).$$

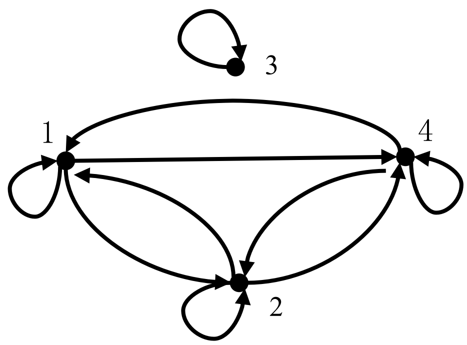
(e) $f(g(y), g(x)) = f(y^2, x^2) = (5x^2 - 1, 7 - y^2)$.

(6’) 9. Let set $X = \{1, 2, 3, 4\}$ and define a relation R on X as follows:
for all $x, y \in X$, $x \, R \, y \Leftrightarrow 3 \mid (x^2 - y^2)$.

(a) Draw the directed graph of R.

(b) Is R an equivalence relation? Explain. If yes, find the distinct equivalence classes of R.

(a)



(b) R is an equivalence relation, because it is reflexive, symmetric, and transitive.
There are two distinct equivalence classes.
$[1] = \{1,2,4\}$
$[3] = \{3\}$

(12’) 10. A CS department has 10 professors, in which 3 are female and 7 are male. We need to form up a committee of 3 professors in this department

(a) How many distinct committees can be formed up?

(b) What is the expected value of the number of female members in this committee?

(c) If two professors are a couple and they are not allowed to be both in the committee, how many distinct committees can be formed up?

(d) If there are 12 available offices in the department building and each of the 10 professors should be assigned exactly one private office, how many distinct office assignments can be made?

(a)

$$\binom{10}{3} = \frac{10!}{3! \cdot (10-3)!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 120$$

(b) The number of distinct committee where zero member is female:

$$\binom{7}{3} = \frac{7!}{3! \cdot (7-3)!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$$

The number of distinct committee where exact one member is female:

$$\binom{3}{1} \cdot \binom{7}{2} = 3 \cdot \frac{7!}{2! \cdot (7-2)!} = 3 \cdot \frac{7 \cdot 6}{2} = 63$$

The number of distinct committee where exact two members are female:

$$\binom{3}{2} \cdot \binom{7}{1} = \frac{3!}{2! \cdot (3-2)!} \cdot 7 = \frac{3 \cdot 2}{2} \cdot 7 = 21$$

The number of distinct committee where exact three member are female:

$$\binom{3}{3} = 1$$

Therefore, the expected value of the number of female members in this committee is

$$0 \cdot \frac{35}{120} + 1 \cdot \frac{63}{120} + 2 \cdot \frac{21}{120} + 3 \cdot \frac{1}{120} = \frac{63+42+3}{120} = \frac{108}{120} = 0.9$$

(c) From the total number of distinct committees, subtract the number of distinct committees where the two professors are both in.

$$\binom{10}{3} - \binom{8}{1} = 120 - 8 = 112$$

(d) The first professor can be in one of the 12 offices, then the second one can be in one of the remaining 11 offices, and the third one can be in one of the remaining 10 offices, etc.

$$\mathrm{P}(12,10) = \frac{12!}{(12-10)!} = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 239500800$$

(10’) 11. Suppose we know that in North Carolina, every 15 out of 1,000 people are UNC fans, every 10 out of 1,000 people are Duke fans, and nobody is both a UNC fan and a Duke fan. Now, we randomly choose people in North Carolina for a survey after the NCAA Men’s Basketball Final 2017.

(a) Suppose we find that 99% of UNC fans watched the Final and 5% of people who are not UNC fans watched the Final. If we randomly choose a person who watched the Final, what is the probability that this person is a UNC fan?

(b) Suppose we find that 99% of UNC fans watched the Final, 50% of Duke fans watched the Final, and 5% of people who are neither UNC fans nor Duke fans watched the Final. If we randomly choose a person who watched the Final, what is the probability that this person is a UNC fan?

(a) Let “W” denote “watched the Final,” and therefore “W^c” denote “did not watch the Final.” Also, let “U” denote “is a UNC fan,” and therefore “U^c” denote “is not a UNC fan.”
Then, we know P(U) = 0.015, P(U^c) = 0.985, P(W | U) = 0.99, P(W | U^c) = 0.05. Therefore,

$$\mathrm{P(U\,|W)} = \frac{\mathrm{P(U \cap W)}}{\mathrm{P(W)}} = \frac{\mathrm{P(W \cap U)}}{\mathrm{P(W \cap U) + P(W \cap U^c)}} = \frac{\mathrm{P(W \mid U) \cdot P(U)}}{\mathrm{P(W \mid U) \cdot P(U) + P(W \mid U^c) \cdot P(U^c)}}$$

$$= \frac{0.99 * 0.015}{0.99 * 0.015 + 0.05 * 0.985} \cong 0.23167 = 23.167\%.$$

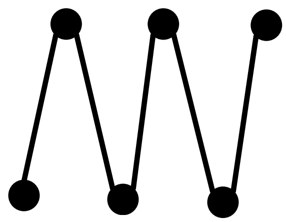
(b) We further let “D” denote “is a Duke fan,” and let “N” denote “is neither a UNC fan nor a Duke fan.” Then, we know that P(D) = 0.01, P(N) = 0.975, P(W | D) = 0.5 and P(W | N) = 0.05. Because nobody is both a UNC fan and a Duke fan, U, D, N are mutually disjoint and the union of these three sets are the set of all people in North Carolina. Therefore,

$$\mathrm{P(U\,|W)} = \frac{\mathrm{P(U \cap W)}}{\mathrm{P(W)}} = \frac{\mathrm{P(W \cap U)}}{\mathrm{P(W \cap U) + P(W \cap D) + P(W \cap N)}}$$

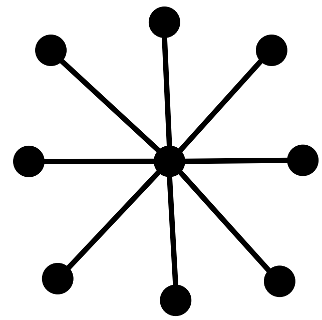
$$= \frac{\mathrm{P(W \mid U) \cdot P(U)}}{\mathrm{P(W \mid U) \cdot P(U) + P(W \mid D) \cdot P(D) + P(W \mid N) \cdot P(N)}}$$

$$= \frac{0.99 * 0.015}{0.99 * 0.015 + 0.5 * 0.01 + 0.05 * 0.975} \cong 0.21647 = 21.647\%.$$

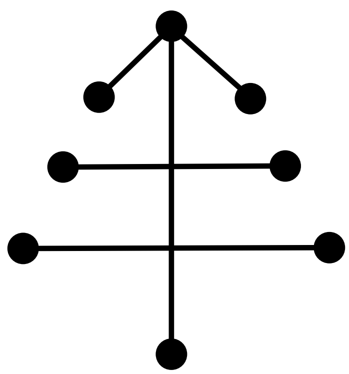
(8') 12. Indicate whether each of the following graphs is a tree.



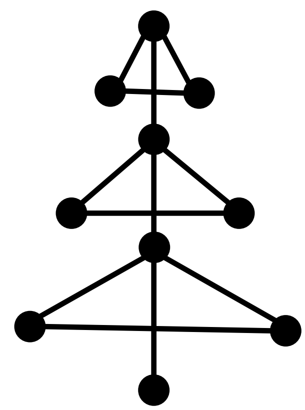
G_1



G_2



G_3



G_4

G_1 is a tree.
G_2 is a tree.
G_3 is not a tree, because it is not connected.
G_4 is not a tree, because there is a cycle.