

Fair Scheduling of Dynamic Task Systems on Multiprocessors*

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Abstract

In dynamic real-time task systems, tasks that are subject to deadlines are allowed to join and leave the system. In previous work, Stoica *et al.* and Baruah *et al.* presented conditions under which such joins and leaves may occur in fair-scheduled uniprocessor systems without causing missed deadlines. In this paper, we extend their work by considering fair-scheduled multiprocessors. We show that their conditions are sufficient on M processors, under all known (dynamic-priority) Pfair scheduling algorithms, if the utilization of every subset of $M - 1$ tasks is at most one. Further, for the general case in which task utilizations are not restricted in this way, we derive sufficient join/leave conditions for the PD² Pfair algorithm. We also show that, in general, these conditions cannot be improved upon without causing missed deadlines.

Keywords: Dynamic task systems, Pfairness, multiprocessors, real-time systems, scheduling

1 Introduction

In many real-time systems, the set of runnable tasks may change dynamically. For example, in an embedded system, different modes of operation may need to be supported; a mode change may require adding new tasks and deleting existing tasks. Another example is a desktop system that supports real-time applications such as multimedia and collaborative-support systems, which may be initiated at arbitrary times.

The distinguishing characteristic of *dynamic* task systems such as these is that tasks are allowed to *join* and *leave* the system. If such joins

and leaves are unrestricted, then the system may become overloaded, and deadlines may be missed. Thus, joins and leaves must be performed only under conditions that ensure that deadline guarantees are not compromised. A suitable join condition usually can be obtained from the feasibility test associated with the scheduling algorithm being used. A leave condition is somewhat trickier. In particular, if an “over-allocated” task is allowed to leave, then it might re-join immediately, and thus effectively execute at a higher-than-prescribed rate.

In this paper, we consider the problem of scheduling such task systems on multiprocessors. This problem has been studied earlier in the context of uniprocessor static-priority [9, 12] and fair-allocation schemes [5, 11]. Our focus is fair scheduling because it is the only known way of optimally scheduling recurrent real-time tasks on multiprocessors [1, 3, 4, 10]. In addition, practical interest in multiprocessor fair scheduling algorithms is growing. For example, Ensim Corp., an Internet service provider, has deployed such algorithms in its product line [8]. Indeed, the need to support dynamic tasks is fundamental in this setting.

In fair scheduling disciplines, tasks are required to make progress at steady rates. Steady allocation rates are ensured by scheduling in a manner that closely tracks an ideal, fluid allocation. The *lag* of a task measures the difference between its ideal and actual allocations. In fair scheduling schemes, lags are required to remain bounded. (Such a bound in turn implies a bound on the timeliness of real-time tasks.) If a task’s lag is positive, then it has been under-allocated; if negative, then it has been over-allocated. In the uniprocessor

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join/leave conditions presented previously [5, 11], a task is allowed to leave iff it is not over-allocated, and join iff the total utilization after it joins is at most one.

Extending the above-mentioned work to multiprocessors is not straightforward; in fact, Baruah *et al.* explicitly noted the multiprocessor case as an open problem [5]. In recent work, dynamic multiprocessor systems were considered by Chandra *et al.* [6, 7]. However, their work was entirely experimental in nature, with no formal analysis of the algorithms considered. In this paper, we present multiprocessor join/leave conditions for which such analysis is provided.

We now briefly describe some of the fair scheduling concepts used in this paper. We then present a more detailed overview of the contributions of this paper.

Pfair scheduling. The *periodic* task model provides the simplest notion of a recurrent real-time task. In this model, successive job releases by the same task are spaced apart by a fixed interval, called the task’s *period*. Periodic tasks can be optimally scheduled on multiprocessors using *Pfair* scheduling algorithms [1, 3, 4]. Pfairness requires the lag of each task to be bounded between -1 and 1 , which is a stronger requirement than periodicity. As we shall see, these lag bounds have the effect of breaking each task into quantum-length *subtasks* that must be scheduled within *windows* of approximately equal lengths. The length and alignment of a task’s windows are determined by its weight. The *weight* or *utilization* of a task is the ratio of its per-job execution cost and period; a task’s weight determines the processor share it requires. Fig. 1(a) shows the subtasks and windows for the first job of a periodic task of weight $8/11$.

In the *sporadic* model, the periodic notion of recurrence is relaxed by specifying a minimum (rather than exact) spacing between consecutive job releases of the same task. In recent work [2, 10], we extended the sporadic model to obtain the *intra-sporadic* (IS) and *generalized intra-sporadic* (GIS) models. The sporadic model allows *jobs* to be released “late”; the IS model allows *subtasks* to be released late, as illustrated in Fig. 1(b). The GIS model is obtained from the IS model by

allowing subtasks to be absent. Fig. 1(c) shows an example.

In [10], we showed that the PD^2 Pfair algorithm optimally schedules static GIS task systems on multiprocessors. In [2], we proved that the (simpler) earliest-pseudo-deadline-first (EPDF) algorithm is optimal for scheduling static IS task systems on two processors. (PD^2 and EPDF are described in Sec. 2.2.)

Contributions. In this paper, we extend our earlier work, as well as prior work on uniprocessor fairness, in several significant ways. First, we show that the previously-presented uniprocessor join/leave conditions [5, 11] are insufficient for avoiding deadline misses when tasks are scheduled using any of a class of algorithms that includes all known (dynamic-priority) Pfair scheduling algorithms. Second, we show that these uniprocessor conditions are sufficient when using any algorithm in this class, if the total weight of any subset of $M - 1$ tasks is at most one at all times. This result extends our earlier result on the optimality of EPDF for two-processor systems [2]. Third, we derive sufficient conditions for the general case in which task weights are not restricted as above, and PD^2 is used for scheduling.

The rest of the paper is organized as follows. In Sec. 2, needed definitions are given. In Sec. 3, our join/leave conditions are stated. Results pertaining to the EPDF and PD^2 algorithms are then presented in Secs. 4 and 5, respectively. We conclude in Sec. 6.

2 Preliminaries

In the following subsections, relevant concepts and terms are defined. We begin with Pfair scheduling.

2.1 Pfair scheduling

In defining notions relevant to Pfair scheduling, we limit attention (for now) to periodic tasks.¹ A periodic task T with an integer *period* $T.p$ and an integer *execution cost* $T.e$ has a *weight* $wt(T) =$

¹Unless specified otherwise, we assume that each periodic task begins execution at time 0.

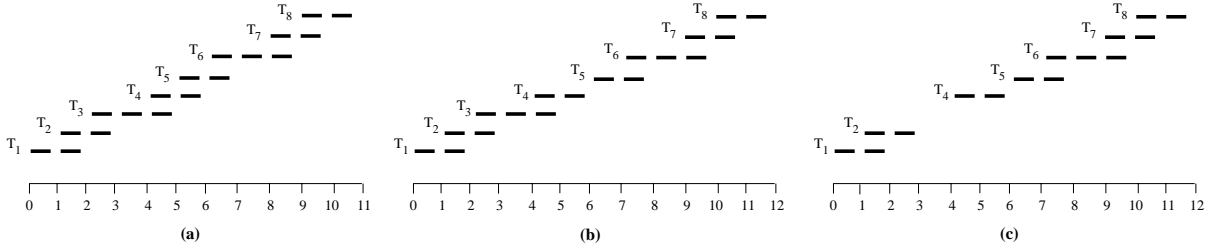


Figure 1: **(a)** Windows of the first job of a periodic task T with weight $8/11$. This job consists of subtasks T_1, \dots, T_8 . Each of these subtasks must be scheduled during its window, or a lag-bound violation will result. (This pattern repeats for every job.) **(b)** The Pfair windows of an IS task. Subtask T_5 becomes eligible one time unit late. **(c)** The Pfair windows of a GIS task. Subtask T_3 is absent and T_5 is one time unit late. (Because T_3 is absent, this is not an IS task.)

$T.e/T.p$, where $0 < wt(T) \leq 1$. A task T is *light* if $wt(T) < 1/2$, and *heavy* otherwise.

Under Pfair scheduling, processor time is allocated in discrete time units, called *quanta*; the time interval $[t, t + 1)$, where t is a nonnegative integer, is called *slot* t . (Hence, time t refers to the beginning of slot t .) In each slot, each processor can be allocated to at most one task. A task may be allocated time on different processors, but not in the same slot (*i.e.*, interprocessor migration is allowed but parallelism is not). The sequence of allocation decisions over time defines a *schedule* S . Formally, $S: \tau \times \mathcal{N} \mapsto \{0, 1\}$, where τ is a set of tasks and \mathcal{N} is the set of nonnegative integers. $S(T, t) = 1$ iff T is scheduled in slot t . Thus, in any M -processor schedule, $\sum_{T \in \tau} S(T, t) \leq M$ for all t .

The notion of a Pfair schedule is defined by comparing such a schedule to a fluid processor-sharing schedule that allocates $wt(T)$ processor time to task T in each slot. Deviance from the fluid schedule is formally captured by the concept of *lag*. The *lag of task T at time t* is $lag(T, t) = wt(T) \cdot t - \sum_{u=0}^{t-1} S(T, u)$. A schedule is *Pfair* iff

$$(\forall T, t :: -1 < lag(T, t) < 1). \quad (1)$$

Informally, the allocation error associated with each task must always be less than one quantum.

The lag bounds above have the effect of breaking each task T into an infinite sequence of *unit-time subtasks*. We denote the i^{th} subtask of task T as T_i , where $i \geq 1$. As in [3], we associate a *pseudo-release* $r(T_i)$ and *pseudo-deadline*² $d(T_i)$ with each

subtask T_i , as follows. (For brevity, we often drop the prefix “pseudo-.”)

$$r(T_i) = \left\lfloor \frac{i-1}{wt(T)} \right\rfloor \wedge d(T_i) = \left\lceil \frac{i}{wt(T)} \right\rceil \quad (2)$$

T_i must be scheduled in the interval $w(T_i) = [r(T_i), d(T_i))$, termed its *window*, or (1) will be violated. Note that $r(T_{i+1})$ is either $d(T_i) - 1$ or $d(T_i)$. Thus, consecutive windows of the same task either overlap by one slot or are disjoint (see Fig. 1(a)). The *length* of T_i ’s window, denoted $|w(T_i)|$, is $d(T_i) - r(T_i)$. As an example, consider subtask T_2 in Fig. 1(a). Here, we have $r(T_2) = 1$, $d(T_2) = 3$, and $|w(T_2)| = 2$. Therefore, T_2 must be scheduled in either slot 1 or 2. (If T_1 is scheduled in slot 1, then T_2 must be scheduled in slot 2.)

2.2 Scheduling Algorithms

In earlier work [2], we proved that the earliest-pseudo-deadline-first (EPDF) Pfair algorithm is optimal on at most two processors, but not on more than two processors. As its name suggests, EPDF gives higher priority to subtasks with earlier deadlines. A tie between subtasks with equal deadlines is broken arbitrarily.

At present, three Pfair scheduling algorithms are known to be optimal on an arbitrary number of processors: PF [3], PD [4], and PD² [1]. These algorithms prioritize subtasks on an EPDF basis, but differ in the choice of tie-breaking rules. PD², which is the most efficient of the three, uses two

²In our earlier work [1, 2, 10], pseudo-deadlines were de-

finned to refer to slots; here, they refer to time. Hence, the formula for $d(T_i)$ given here is slightly different.

tie-break parameters.

The first PD² tie-break is a bit, denoted by $b(T_i)$. As mentioned earlier, consecutive windows of a task are either disjoint or overlap by one slot. $b(T_i)$ distinguishes between these two possibilities.

$$b(T_i) = d(T_i) - r(T_{i+1}) \quad (3)$$

For example, in Fig. 1(a), $b(T_i) = 1$ for $1 \leq i \leq 7$ and $b(T_8) = 0$. PD² favors a subtask with a b -bit of 1 over one with a b -bit of 0. Informally, it is better to execute T_i “early” if its window overlaps that of T_{i+1} , because this potentially leaves more slots available to T_{i+1} .

The second PD² tie-break, the *group deadline*, is needed in systems containing tasks with windows of length two. A task T has such windows iff $1/2 \leq wt(T) < 1$. Consider a sequence T_i, \dots, T_j of subtasks of such a task T such that $b(T_k) = 1 \wedge |w(T_{k+1})| = 2$ for all $i \leq k < j$. Scheduling T_i in its last slot forces the other subtasks in this sequence to be scheduled in their last slots. For example, in Fig. 1(a), scheduling T_3 in slot 4 forces T_4 and T_5 to be scheduled in slots 5 and 6, respectively. The group deadline of a subtask T_i , denoted $D(T_i)$, is the earliest time by which such a “cascade” must end. Formally, it is the earliest time t , where $t \geq d(T_i)$, such that either $(t = d(T_k) \wedge b(T_k) = 0)$ or $(t+1 = d(T_k) \wedge |w(T_k)| = 3)$ for some subtask T_k . For example, in Fig. 1(a), $D(T_3) = d(T_6) - 1 = 8$ and $D(T_7) = d(T_8) = 11$. PD² favors subtasks with later group deadlines because *not* scheduling them can lead to longer cascades, which places more constraints on the future schedule.

We can now describe the PD² priority definition. If subtasks T_i and U_j are both eligible at time t , then PD² prioritizes T_i over U_j at t if $(d(T_i) < d(U_j))$, or $(d(T_i) = d(U_j) \wedge b(T_i) = 1 \wedge b(U_j) = 0)$, or $(d(T_i) = d(U_j) \wedge b(T_i) = b(U_j) \wedge D(T_i) \geq D(U_j))$. (Refer to [1] for a more detailed explanation of PD².)

2.3 Generalized Intra-sporadic Tasks

Having described the concept of Pfair scheduling, we now describe the intra-sporadic (IS) and the generalized intra-sporadic (GIS) task models.

The IS model generalizes the sporadic model by

allowing separation between consecutive subtasks of a task. More specifically, the separation between $r(T_i)$ and $r(T_{i+1})$ is allowed to be more than $\lfloor i/wt(T) \rfloor - \lfloor (i-1)/wt(T) \rfloor$, which is the separation if T were periodic. Thus, an IS task is obtained by allowing a task’s windows to be right-shifted from where they would appear if the task were periodic. Fig. 1(b) illustrates this.

Each subtask of an IS task has an *offset* that gives the amount by which its window has been right-shifted. Let $\theta(T_i)$ denote the offset of subtask T_i . Then, by (2), we have the following.

$$r(T_i) = \theta(T_i) + \left\lfloor \frac{i-1}{wt(T)} \right\rfloor \quad (4)$$

$$d(T_i) = \theta(T_i) + \left\lceil \frac{i}{wt(T)} \right\rceil \quad (5)$$

These offsets are constrained so that the separation between any pair of subtask releases is at least the separation between those releases if the task were periodic. Formally, the offsets satisfy the following property.

$$k \geq i \Rightarrow \theta(T_k) \geq \theta(T_i) \quad (6)$$

Each subtask T_i has an additional parameter $e(T_i)$ that corresponds to the first time slot in which T_i is eligible to be scheduled. It is assumed that $e(T_i) \leq r(T_i)$ and $e(T_i) \leq e(T_{i+1})$ for all $i \geq 1$. Allowing $e(T_i)$ to be less than $r(T_i)$ is equivalent to allowing “early” subtask releases as in ERfair scheduling [1]. The interval $[r(T_i), d(T_i))$ is called the *PF-window* of T_i , while the interval $[e(T_i), d(T_i))$ is called the *IS-window* of T_i .

The GIS model generalizes the IS model by allowing subtasks to be absent. Thus, the subtasks of a GIS task are a subset of the subtasks of an IS task. Fig. 1(c) shows an example. The formulae for subtask release times and deadlines of a GIS task are the same as for an IS task. The b -bit and group deadline for a subtask are defined as before assuming that future subtask releases are as early as possible. Thus, $D(T_i) = \theta(T_i) + D_p(T_i)$, where $D_p(T_i)$ is T_i ’s group deadline if T were periodic.

Because the GIS model generalizes the other task models above, it is the notion of recurrence considered hereafter. We now present some definitions and properties about GIS task systems.

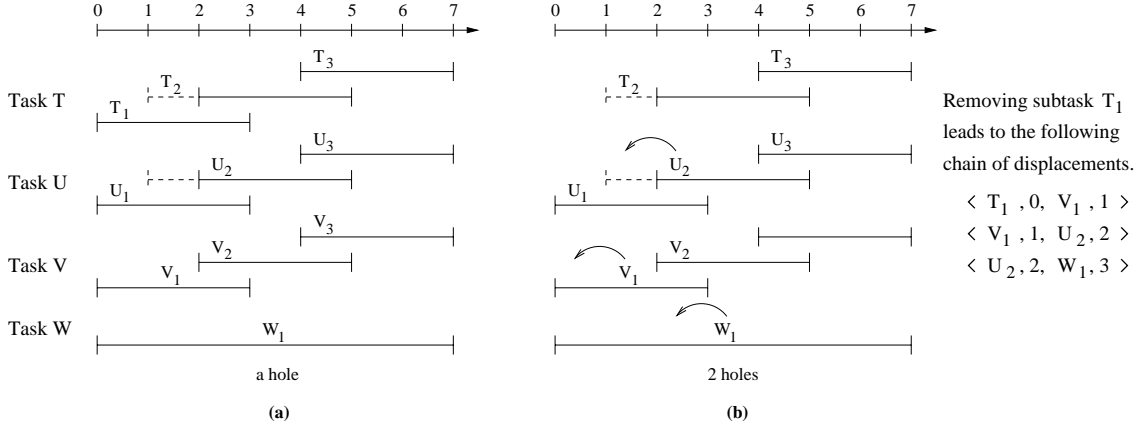


Figure 2: A schedule for three tasks of weight $3/7$ and one task of weight $1/7$ on two processors. Solid lines depict PF-windows; dashed lines are used to show the extent to which an IS-window extends before a corresponding PF-window. Note that only subtasks T_2 and U_2 are eligible before their PF-windows. Inset (b) illustrates the displacements caused by the removal of subtask T_1 from the schedule shown in inset (a).

Terminology. An *instance* of a task system is obtained by specifying a unique assignment of release times and eligibility times for each subtask, subject to (6). Note that the deadline of a subtask is automatically determined once its release time is fixed (refer to (4) and (5)). If a task T , after executing subtask T_i , releases subtask T_k , then T_k is called the *successor* of T_i and T_i is called the *predecessor* of T_k (e.g., T_4 is T_2 's successor in Fig. 1(c)).

Feasibility. In [2, 10], we showed that a GIS task system τ is feasible on M processors iff

$$\sum_{T \in \tau} wt(T) \leq M. \quad (7)$$

In fact, the proof of this shows that a schedule exists in which each subtask is scheduled in its PF-window. In [10], we also proved that PD² correctly schedules any GIS task system for which (7) holds.

Displacements. By definition, the removal of a subtask from one instance of a GIS task system results in another valid instance. Let $X^{(i)}$ denote a subtask of any task in a GIS task system τ . Let S denote any schedule of τ obtained by an EPDF-based algorithm. Assume that removing $X^{(1)}$ scheduled at slot t_1 in S causes $X^{(2)}$ to shift from slot t_2 to t_1 , where $t_1 \neq t_2$, which in turn may cause other shifts. We call this shift a *displacement* and represent it

by a four-tuple $\langle X^{(1)}, t_1, X^{(2)}, t_2 \rangle$. A displacement $\langle X^{(1)}, t_1, X^{(2)}, t_2 \rangle$ is *valid* iff $e(X^{(2)}) \leq t_1$. Because there can be a cascade of shifts, we may have a *chain* of displacements, as illustrated in Fig. 2.

Removing a subtask may also lead to slots in which some processors are idle. In a schedule S , if k processors are idle in slot t , then we say that there are k *holes* in S in slot t . Note that holes may exist because of late subtask releases, even if total utilization is M .

The lemmas below concern displacements and holes. The first two were proved earlier for PD² [10] but apply to all algorithms that prioritize subtasks on an EPDF basis. Lemma 1 states that a subtask removal can only cause left-shifts, as in Fig. 2(b). Lemma 2 indicates when a left-shift into a slot with a hole can occur. Lemma 3 shows that shifts across a hole cannot occur. Here, τ is an instance of a GIS task system and S denotes a schedule for τ obtained by an EPDF-based algorithm. Throughout this paper, we assume that ties among subtasks are resolved consistently, *i.e.*, if τ' is obtained from τ by a subtask removal, then the relative priorities of two subtasks in τ' are the same as in τ .

Lemma 1 *Let $X^{(1)}$ be a subtask that is removed from τ , and let the resulting chain of displacements in S be $C = \Delta_1, \Delta_2, \dots, \Delta_k$, where $\Delta_i = \langle X^{(i)}, t_i, X^{(i+1)}, t_{i+1} \rangle$. Then $t_{i+1} > t_i$ for all $i \in \{1, \dots, k\}$.*

Lemma 2 Let $\Delta = \langle X^{(1)}, t_1, X^{(2)}, t_2 \rangle$ be a valid displacement in S . If $t_1 < t_2$ and there is a hole in slot t_1 in that schedule, then $X^{(2)}$ is $X^{(1)}$'s successor in τ .

Lemma 3 Let $\Delta = \langle X^{(1)}, t_1, X^{(2)}, t_2 \rangle$ be a valid displacement in S . If $t_1 < t_2$ and there is a hole in slot t' such that $t_1 \leq t' < t_2$ in that schedule, then $t' = t_1$ and $X^{(2)}$ is the successor of $X^{(1)}$ in τ .

Proof (of Lemma 3): Since Δ is valid, $e(X^{(2)}) \leq t_1$. If $t_1 < t'$, then $e(X^{(2)}) < t'$, implying that $X^{(2)}$ is not scheduled in slot $t_2 > t'$, as assumed, since there is a hole in t' . Thus, $t_1 = t'$; by Lemma 2, $X^{(2)}$ is $X^{(1)}$'s successor. \square

Flows and lags in GIS task systems. Let v be the time at which a GIS task T joins the system. The lag of T at time t is defined in the same way as it is defined for periodic tasks. Let $ideal(T, v, t)$ denote the share that T receives in a fluid schedule in $[v, t)$. Then,

$$lag(T, t) = ideal(T, v, t) - \sum_{u=v}^{t-1} S(T, u). \quad (8)$$

Before defining $ideal(T, v, t)$, we define $flow(T, u)$, which is the share assigned to task T in slot u . $flow(T, u)$ is defined in terms of a function f that indicates the share assigned to each subtask in each slot. Define $f(T_i, u)$ as follows.

$$\begin{cases} \left(\left\lfloor \frac{i-1}{wt(T)} \right\rfloor + 1 \right) \cdot wt(T) - (i-1), & \text{if } u = r(T_i) \\ i - \left(\left\lfloor \frac{i}{wt(T)} \right\rfloor - 1 \right) \cdot wt(T), & \text{if } u = d(T_i) - 1 \\ wt(T), & \text{if } r(T_i) < u < d(T_i) - 1 \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

Fig. 3 shows the values of f for different subtasks of a task of weight $5/16$. $flow(T, u)$ is simply defined as $flow(T, u) = \sum_i f(T_i, u)$. Observe that $flow(T, u)$ usually equals $wt(T)$, but in certain slots, it may be less than $wt(T)$, so that each subtask of T has a unit share. In [10], we proved the following flow properties.

(F1) For all time slots t , $flow(T, t) \leq wt(T)$.

(F2) Let T_i be a subtask of a GIS task and let T_k be its successor. If $b(T_i) = 1$ and

$$r(T_k) \geq d(T_i), \text{ then } flow(T, d(T_i) - 1) + flow(T, d(T_i)) \leq wt(T).$$

For example, in Fig. 3(b), $flow(T, 3) + flow(T, 4) = 1/16 < 5/16$ and $flow(T, 14) + flow(T, 15) = 5/16$.

Given the above flow values, $ideal(T, v, t)$ is defined as $\sum_{u=v}^{t-1} flow(T, u)$. Hence, by (8), we obtain that $lag(T, t+1) = \sum_{u=0}^t (flow(T, u) - S(T, u)) = lag(T, t) + flow(T, t) - S(T, t)$. Similarly, the total lag for a schedule S and task system τ at time $t+1$, denoted by $LAG(\tau, t+1)$, is

$$\begin{aligned} LAG(\tau, t+1) &= LAG(\tau, t) \\ &+ \sum_{T \in \tau} (flow(T, t) - S(T, t)). \end{aligned} \quad (10)$$

($LAG(\tau, 0)$ is defined to be 0.) The lemma below (proved in [10]) is used in the following sections.

Lemma 4 If $LAG(\tau, t) < LAG(\tau, t+1)$, then there is a hole in slot t .

Proof Sketch: Suppose there is no hole in slot t . Then, $\sum_{T \in \tau} S(T, t) = M$. On the other hand, by (F1) and (7), $\sum_{T \in \tau} flow(T, t) \leq M$. Therefore, by (10), LAG cannot increase from t to $t+1$. \square

3 Dynamic Task Systems

Prior work in the real-time-systems literature has focused mostly on static systems, in which the set of tasks does not change with time. However, systems exist in which the set of tasks may change frequently. One example of such a system is a virtual-reality application in which the user moves within a virtual environment. As the user moves and the virtual scene changes, the time required to render the scene may vary substantially. If a single task is responsible for rendering, then its weight may change frequently. Task *reweighting* can be modeled as a leave-and-join problem, in which a task with the old weight leaves and a task with the new weight joins.

As shown in [2, 10], a valid schedule can be obtained for any *static* GIS task system satisfying (7) by constructing a flow network with a real-valued flow based on the $flow$ values defined in Sec. 2 and illustrated in Fig. 3. A corresponding integral flow exists because all edge capacities in the network

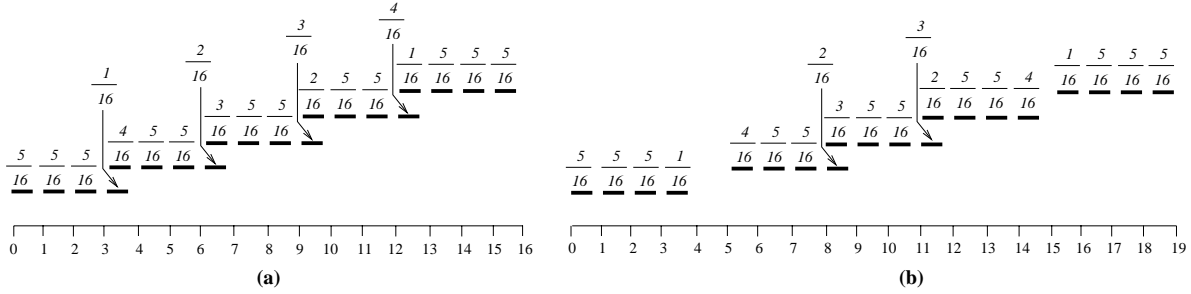


Figure 3: Fluid schedule for a task T of weight $5/16$. The share of each subtask in the slots of its window is shown. In (a), no subtask is released late; in (b), T_2 and T_3 are released late. Note that $flow(T, 3)$ is either $5/16$ or $1/16$ depending on when subtask T_2 is released.

are integers. This gives us a Pfair schedule. This argument can be easily extended to apply to any *dynamic* task system for which the total utilization of all tasks present at every instant is at most M . This proof produces an offline schedule³ in which *each subtask is scheduled in its PF-window*.

A condition for allowing tasks to join the system is an immediate consequence of this feasibility test, *i.e.*, admit a task if the total utilization is at most M after its admission. The important question left is — *when should a task be allowed to leave the system?* (Here, we are referring to the time when we can reclaim the utilization of the task. The task may be actually allowed to leave the system earlier.) As shown in [5, 11], if an over-allocated task is allowed to leave, then it can re-join immediately and effectively execute at a rate higher than its specified rate causing other tasks to miss their deadlines. Hence, we only allow non-over-allocated tasks (*i.e.*, tasks with non-negative lags) to leave the system, as stated in (C1) below.

- (C1) *Join condition:* A task T can join at time t iff the total utilization after joining is at most M . If T joins at time t , then $\theta(T_1)$ is set to t .
Leave condition: A task T can leave at time t iff $t \geq d(T_i)$, where T_i is the last-released subtask of T .

The condition $t \geq d(T_i)$ implies that $lag(T, t) = 0$. To see why, note that since T_i is the last-released

subtask of T , T is neither under-allocated nor over-allocated at time $d(T_i)$. Thus, only tasks with zero lag are allowed to leave the system. It is easy to extend (C1) to allow a task with positive lag to leave. This is because such a task is under-allocated, and hence its last-released subtask has not yet been scheduled. Intuitively, not scheduling a subtask is equivalent to removing it, and by Lemma 1, the removal of a subtask cannot lead to a missed deadline. Thus, we can allow task T to leave the system if (C1) is satisfied by T_{i-1} , *i.e.*, the last-scheduled subtask of T . However, for simplicity, we assume (C1) as stated above in our proof.

(C1) is a direct extension of the uniprocessor conditions presented by Baruah *et al.* [5] and Stolica *et al.* [11]. However, as shown below, it can cause missed deadlines and hence, is not sufficient on multiprocessors.

The theorem below applies to any “weight-consistent” Pfair scheduling algorithm. An algorithm is *weight-consistent* if, given two tasks T and U of equal weight with eligible subtasks T_i and U_j , respectively, where $i = j$ and $r(T_i) = r(U_j)$ (and hence, $d(T_i) = d(U_j)$), T_i has priority over a third subtask V_k iff U_j does. All known (dynamic-priority) Pfair scheduling algorithms are weight-consistent.

Theorem 1 *No weight-consistent scheduler can guarantee all deadlines on multiprocessors under (C1).*

Proof: Consider a class of task systems consisting of two sets of tasks X and Y of weights $w_1 = 2/5$ and $w_2 = 3/8$, respectively. Let X_f (Y_f) denote

³The schedule is offline because all subtask release times must be known beforehand. An online algorithm that does not respect a fixed quantum can also be easily obtained from the *flow* values.

the set of first subtasks of tasks in X (Y). We construct a task system depending on the task weight favored by the scheduler. We say that X_f is favored (analogously for Y_f) if, whenever subtasks in X_f and Y_f are released at the same time, those in X_f are favored.

Case 1: X_f is favored. Consider a dynamic task system consisting of the following types of tasks to be scheduled on 15 processors. (In each of our counterexamples, no subtask is eligible before its PF-window.)

Type A: 8 tasks of weight w_2 that join at time 0.

Type B: 30 tasks of weight w_1 that join at time 0 and leave at time 3.

Type C: 30 tasks of weight w_1 that join at time 3.

Because $30w_1 + 8w_2 = 15$, this task system is feasible, and the join condition for type-C tasks in (C1) is satisfied. Note that $d(T_1) = \lceil \frac{5}{2} \rceil = 3$ for every type-B task T ; hence, the leave condition in (C1) is also satisfied.

Since subtasks in X_f are favored, type-B tasks are favored over type-A tasks at times 0 and 1. Hence, the schedule for $[0, 3)$ will be as shown in Fig. 4(a). Consider the interval $[3, 8)$. Each type-A task has two subtasks remaining for execution, which implies that the type-A tasks need 16 quanta. Similarly, each type-C task also has two subtasks, which implies that the type-C tasks need 60 quanta. However, the total number of quanta in $[3, 8)$ is $15 \cdot (8 - 3) = 75$. Thus, one subtask will miss its deadline at or before time 8.

Case 2: Y_f is favored. Consider a dynamic task system consisting of the following types of tasks to be scheduled on 8 processors.

Type A: 5 tasks of weight w_1 that join at time 0.

Type B: 16 tasks of weight w_2 that join at time 0 and leave at time 3.

Type C: 16 tasks of weight w_2 that join at time 3.

Because $5w_1 + 16w_2 = 8$, this task system is feasible, and the join condition for type-C tasks in (C1) is satisfied. Note that $d(T_1) = \lceil \frac{8}{3} \rceil = 3$ for every type-B task T ; hence, the leave condition in (C1) is also satisfied.

Since subtasks in Y_f are favored, type-B tasks are favored over type-A tasks at times 0 and 1. Hence, the schedule for $[0, 3)$ will be as shown in Fig. 4(b). Consider the interval $[3, 35)$. The number of subtasks of each type-A task that need to be executed in $[3, 35)$ is $1 + (35 - 5) \cdot 2/5 = 13$. Similarly, the number of subtasks of each type-C task is $(35 - 3) \cdot 3/8 = 12$. The total is $5 \cdot 13 + 16 \cdot 12 = 257$, whereas the number of quanta in $[3, 35)$ is $(35 - 3) \cdot 8 = 256$. Thus, one subtask will miss its deadline at or before time 35. \square

Theorem 1 can be “circumvented” if it can be known at the time a subtask is *released* whether it is the final subtask of its task. For example, in Fig. 4(a), if we knew that the first subtask T_1 of each type-B task is its last, then we could have given T_1 an effective b -bit of zero. Hence, PD^2 would have scheduled it with a lower priority than any type-A task. However, in general, such knowledge may not be available to the scheduler.

The examples in Fig. 4(a)–(b) show that allowing a light task T to leave at $d(T_i)$ when $b(T_i) = 1$ can lead to deadline misses. We now derive a similar, but stronger, condition for heavy tasks.

Theorem 2 *If a heavy task T is allowed to leave before $D(T_i)$, where T_i is the last-released subtask of T , then there exist task systems that miss a deadline under PD^2 .*

Proof: Consider the following dynamic task system to be scheduled on 35 processors, where $2 \leq t \leq 4$.

Type A: 9 tasks of weight $7/9$ that join at time 0.

Type B: 35 tasks of weight $4/5$ that join at time 0 and leave at time t ; each releases one subtask.

Type C: 35 tasks of weight $4/5$ that join at time t .

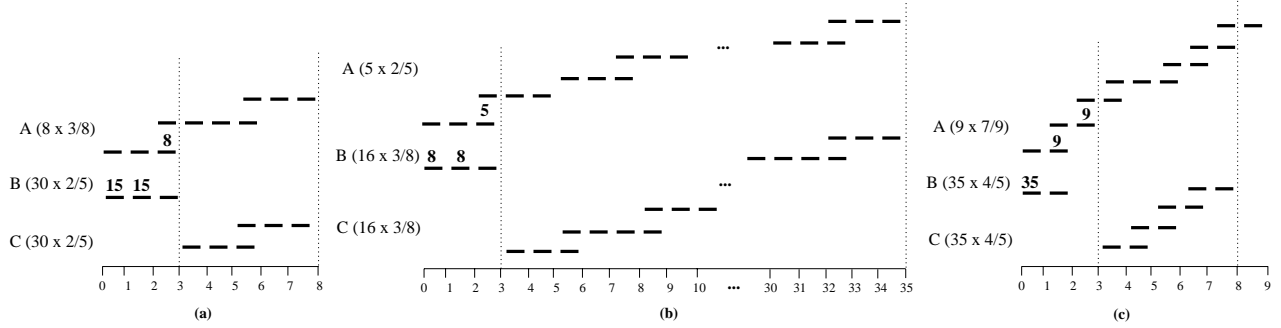


Figure 4: Counterexamples demonstrating insufficiency of (C1) and tightness of (C2). An integer value n in slot t of some window means that n of the subtasks that must execute within that window are scheduled in slot t . No integer value means that no such subtask is scheduled in slot t . The vertical lines depict intervals with excess demand. (a) Theorem 1. Case 1: Tasks of weight $2/5$ are favored at times 0 and 1. (b) Theorem 1. Case 2: Tasks of weight $3/8$ are favored at times 0 and 1. (c) Theorem 2. The tasks of weight $4/5$ are allowed to leave at time 3 and re-join immediately.

All type-A and type-B tasks have the same PD² priority at time 0, because each has a deadline at time 2, a b -bit of 1, and a group deadline at time 5. Hence, the type-B tasks may be given higher priority.⁴ Assuming this, Fig. 4(c) depicts the schedule for the case of $t = 3$.

Consider the interval $[t, t + 5)$. Each type-A and type-C task has four subtasks with deadlines in $[t, t + 5)$ (see Fig. 4(c)). Thus, $9 \cdot 4 + 35 \cdot 4 = 35 \cdot 5 + 1$ subtasks must be executed in $[t, t + 5)$. Since $35 \cdot 5$ quanta are available in $[t, t + 5)$, one subtask will miss its deadline. \square

Although (C1) is not sufficient in general, in Sec. 4, we show that it is sufficient for a restricted class of task systems, even under EPDF. Theorems 1 and 2 provide us with the following new conditions, which are sufficient when used with PD² (proved in Sec. 5). Theorems 1 and 2 also show that these are tight.

(C2) Join condition: A task T can join at time t iff the total utilization after joining is at most M . If T joins at time t , then $\theta(T_1)$ is set to t . **Leave condition:** Let T_i denote the last-released subtask of T . If T is light, then T can leave at time t iff either $t = d(T_i) \wedge b(T_i) = 0$ or $t > d(T_i)$ holds. If T is heavy, then T can leave at time t iff $t \geq D(T_i)$.

⁴This counterexample applies to PF and PD as well, because both favor the type-B tasks at time 0.

As before with (C1), when a task T leaves the system at time t , (C2) implies that $lag(T, t) = 0$. We can also allow T to leave with positive lag, provided its last-scheduled subtask satisfies the leave condition in (C2).

(C2) reduces to (C1) if the leaving task is actually periodic or sporadic and its final job executes for its worst-case execution requirement. That is, such a task can leave at the deadline of its last job. To see why, note that if T_i is the last subtask of the final job, then, because consecutive task periods do not overlap, $b(T_i) = 0$ (and hence $D(T_i) = d(T_i)$, if T is heavy). Thus, by (C2), T can leave at time $d(T_i)$, as in (C1).

4 Sufficiency of (C1) for Restricted Systems

In this section, we show that task systems can be correctly scheduled using EPDF on M processors provided (C1) and (M1) below hold.

(M1) At any time, the sum of the weights of the $M - 1$ heaviest tasks is at most 1.

We use the phrase “C1M1 task system” to refer to task systems satisfying both (C1) and (M1). Assume to the contrary that there exists a C1M1 task system τ that misses a deadline under EPDF. Let S denote its EPDF schedule. Let T_i be the subtask (in some given schedule) with the earliest deadline

among all subtasks that miss a deadline, and let $t_d = d(T_i)$. Thus, all subtasks with deadlines less than t_d meet their deadlines.

Note that any subtask with deadline after t_d is scheduled at a slot prior to t_d only if no subtask with a deadline at most t_d is eligible at that slot. Thus, the scheduling of T_i is not affected by subtasks with deadlines greater than t_d . Hence, we can assume that no task in τ releases any subtask with a deadline greater than t_d . In other words,

$$\text{for every subtask } U_j \in \tau, d(U_j) \leq t_d. \quad (11)$$

Using this, we obtain the following lower bound on $LAG(\tau, t_d)$.

Lemma 5 $LAG(\tau, t_d) \geq 1$.

Proof: By (10), we have

$$LAG(\tau, t_d) = \sum_{t=0}^{t_d-1} \sum_{T \in \tau} \text{share}(T, t) - \sum_{t=0}^{t_d-1} \sum_{T \in \tau} S(T, t).$$

The first term on the right-hand side of the above equation is the total share in the ideal schedule in $[0, t_d)$, which equals the total number of subtasks in τ . (Follows from (11).) The second term corresponds to the number of subtasks scheduled by EPDF in $[0, t_d)$. Since T_i misses its deadline at t_d , the difference between these two terms is at least one. \square

Because $LAG(\tau, 0) = 0$, it follows by Lemma 5 that there exists a time $t < t_d$ such that $LAG(\tau, t) < 1$ and $LAG(\tau, t+1) \geq 1$. We now prove some properties about task lags at time $t+1$; using these properties and (M1), we later derive a contradiction concerning the existence of time t .

By Lemma 4, there is at least one hole in slot t (i.e., in $[t, t+1)$). In other words, the number of tasks scheduled in slot t is at most $M-1$. Let A denote the set of tasks scheduled in slot t . Then, we have

$$|A| \leq M - 1. \quad (12)$$

Let B denote the set of tasks not in A that are “active” at t . A task U is *active* at time t if it has a subtask U_j such that $e(U_j) \leq t < d(U_j)$. (A task may be inactive either because it has already left the system or because of a late subtask release.)

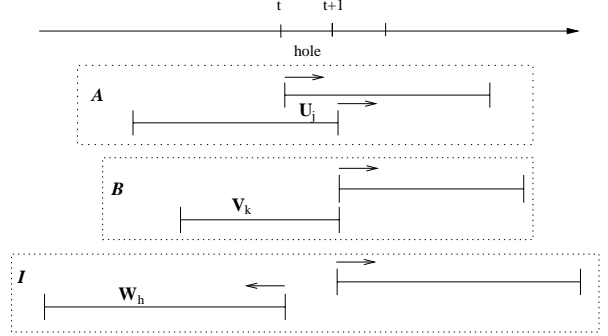


Figure 5: In this figure, PF-windows are denoted by line segments. An arrow over a release (respectively, deadline) indicates that the release (respectively, deadline) could be anywhere in the direction of the arrow. There is a hole in slot t . (a) Sets A , B , and I . The PF-windows of a sample task of each set are shown.

Consider any task $U \in B$ and let U_j be such that $e(U_j) \leq t < d(U_j)$. Because there is a hole in slot t and no subtask of U is scheduled at time t , and because $e(U_j) \leq t < d(U_j)$, U_j must be scheduled before time t .

Let I denote the set of the remaining tasks that are not active at time t . Fig. 5 shows how the tasks in A , B , and I are scheduled. We now estimate the *lag* values for the tasks in each of A , B , and I at time $t+1$.

Lemma 6 For $W \in I$, $lag(W, t+1) = 0$.

Proof: Consider any subtask W_h of task W . If $e(W_h) \geq t+1$, then $r(W_h) \geq t+1$. Therefore, by (9), $f(W_h, u) = 0$ for all slots $u \leq t < r(W_h)$. Hence, the share of W_h in the ideal schedule in $[0, t+1)$ is zero. Also, in the EPDF schedule, W_h is scheduled at or after $t+1$. On the other hand, if $e(W_h) \leq t$, then by the definition of I , $d(W_h) \leq t < t_d$. Since such a subtask meets its deadline, W_h is scheduled in $[0, t)$. Hence, the share received by W_h in $[0, t+1)$ is one in both the ideal and EPDF schedules. Thus, we have shown that for all subtasks of W , the share of that subtask in $[0, t+1)$ is the same in both the ideal and EPDF schedules. (Furthermore, recall from Sec. 3 that any task that leaves the system does so with zero lag.) Therefore, $lag(W, t+1) = 0$. \square

Lemma 7 For $V \in B$, $lag(V, t+1) \leq 0$.

Proof: Consider any subtask V_k of task V . Again, as in the proof of Lemma 6, if $r(V_k) \geq t + 1$, then by (9), the share of V_k in $[0, t + 1)$ in the ideal schedule is zero. On the other hand, if $r(V_k) \leq t$, then, as discussed earlier, V_k is scheduled before t because of the hole in slot t . Thus, the share of V_k in $[0, t + 1)$ is one in the EPDF schedule, and at most one in the ideal schedule. ($d(V_k)$ may be greater than $t + 1$, in which case a portion of V_k 's share in the ideal schedule is allocated after $t + 1$.) Thus, for any subtask of V , its share in $[0, t + 1)$ in the EPDF schedule is at least its share in $[0, t + 1)$ in the ideal schedule. Hence, $lag(V, t + 1) \leq 0$. \square

Lemma 8 For $U \in A$, $lag(U, t + 1) < wt(U)$.

Proof: Let U_j be the subtask of U scheduled at time t . Since $t < t_d$, U_j meets its deadline. Therefore, $d(U_j) \geq t + 1$.

If $d(U_j) > t + 1$, then by (4)–(6), $r(U_{j+1}) \geq t + 1$. Reasoning exactly as in the proof of Lemma 7, we can show that $lag(U, t + 1) \leq 0$.

In the rest of the proof, we assume that $d(U_j) = t + 1$. Let U_k be U_j 's successor (if it exists). Since consecutive PF-windows overlap by at most one slot, $r(U_k) \geq d(U_j) - 1$, *i.e.*, $r(U_k) \geq t$.

If $r(U_k) \geq t + 1$, then the share in the ideal schedule for any subtask after U_j is zero. Thus, the total share of U in $[0, t + 1)$ is same in both the ideal and the EPDF schedules, *i.e.*, $lag(U, t + 1) = 0$.

If $r(U_k) = t$, then $r(U_k) = d(U_j) - 1$. Therefore, by (4)–(6), we obtain $k = j + 1$. Therefore, by (4) and (5), $\lfloor j/wt(U) \rfloor = \lceil j/wt(U) \rceil - 1$, which implies that $\lfloor j/wt(U) \rfloor < j/wt(U)$.

The excess share of U in the ideal schedule in $[0, t + 1)$ is due to $f(U_k, t)$. Therefore, we have $lag(U, t + 1) \leq f(U_k, t)$, *i.e.*, $lag(U, t + 1) \leq f(U_{j+1}, r(U_{j+1}))$.

By (9), $f(U_{j+1}, r(U_{j+1})) = (\lfloor j/wt(U) \rfloor + 1) \cdot wt(U) - j$. Hence, $lag(U, t + 1) \leq (\lfloor j/wt(U) \rfloor + 1) \cdot wt(U) - j < (j/wt(U) + 1) \cdot wt(U) - j$. Thus, $lag(U, t + 1) < wt(U)$. \square

Because $LAG(\tau, t + 1) = \sum_{U \in A \cup B \cup I} lag(U, t + 1)$, by Lemmas 6–8, $LAG(\tau, t + 1) < \sum_{U \in A} wt(U)$. By (12), $|A| \leq M - 1$. Therefore, by (M1), $LAG(\tau, t + 1) < 1$, contradicting our assumption about t . Thus, we have the following theorem.

Theorem 3 EPDF correctly schedules every C1M1 task system on M processors.

Ensuring (M1) involves identifying the $M - 1$ heaviest tasks and summing their weights. A more efficient (and more restrictive) way to enforce (M1) is to require each individual task weight to be at most $1/(M - 1)$.

Corollary 1 EPDF correctly schedules any dynamic GIS task system satisfying (C1) on $M (> 1)$ processors if the weight of each task is at most $1/(M - 1)$.

Since any feasible *static* task system satisfies (C1), we obtain the following generalization of an earlier result that EPDF is optimal for scheduling IS tasks on two processors [2].

Corollary 2 EPDF correctly schedules any feasible static GIS task system satisfying (M1) on M processors.

Condition (M1) can be improved by more accurately bounding $lag(U, t + 1)$ for $U \in A$. Let $U.f = \frac{U.e - gcd(U.e, U.p)}{U.p}$. Using (9), it can be shown that if $d(U_j) = r(U_{j+1}) + 1$, then $U.f$ is the maximum share subtask U_{j+1} can have in slot $r(U_{j+1})$. Thus, we can improve Lemma 8 to show that $lag(U, j + 1) \leq U.f$ for $U \in A$. Performing the same analysis as above, we obtain a contradiction if $\sum_{U \in A} U.f < 1$. Thus, EPDF produces a correct schedule if, at all times, $\sum_{U \in H} U.f < 1$ for all sets H of at most $M - 1$ tasks.

5 Sufficiency of (C2) for PD²

We now prove by contradiction that PD² correctly schedules any dynamic task system for which (C2) holds. The proof strategy used here is similar to that used in [10].

Suppose that PD² misses a deadline for some task system that satisfies (C1). Then there exists a time t_d and a task system τ as given in Defs. 1 and 2 below.

Definition 1 t_d is the earliest time at which any task system instance misses a deadline under PD². \square

Definition 2 τ is an instance of a task system with the following properties.⁵

(T1) τ misses a deadline under PD² at t_d .

(T2) No task system instance satisfying (T1) releases fewer subtasks in $[0, t_d)$ than τ .

(T3) No task system instance satisfying (T1) and (T2) has a larger rank than τ , where the *rank* of an instance is the sum of the eligibility times of all subtasks with deadlines at most t_d . \square

By (T1), (T2), and Def. 1, exactly one subtask in τ misses its deadline: if several subtasks miss their deadlines, all but one can be removed and the remaining subtask will still miss its deadline, contradicting (T2).

We now prove several properties about the S , the PD² schedule for τ .

Lemma 9 *The following properties hold for τ and S , the EPDF schedule for τ , where T_i is any subtask in S .*

(a) *Let t be the time at which T_i is scheduled. Then, $e(T_i) \geq \min(r(T_i), t)$.*

(b) *Let t be as in (a). If either $d(T_i) > t + 1$ or $d(T_i) = t + 1 \wedge b(T_i) = 0$, then T_i 's successor is not eligible before $t + 1$.*

(c) *For all T_i , $d(T_i) \leq t_d$.*

(d) *There are no holes in slot $t_d - 1$.*

(e) $LAG(\tau, t_d) = 1$.

(f) $LAG(\tau, t_d - 1) \geq 1$.

Proof of (a): Suppose that $e(T_i) < \min(r(T_i), t)$. Consider the task system instance τ' obtained from τ by changing $e(T_i)$ to $\min(r(T_i), t)$. Note that $e(T_i)$ is still at most $r(T_i)$ and τ' 's rank is larger than τ 's. It is easy to show that the relative priorities of the subtasks do not change for any slot $u \in \{0, \dots, t_d - 1\}$, and hence, τ' and τ have identical PD² schedules. Thus, τ' misses a deadline at t_d , contradicting (T3).

⁵Note that (T1)–(T3) are being applied *in sequence*; e.g., τ is not claimed to be of maximal rank — rather, its rank is maximal *among those task system instances satisfying (T1) and (T2)*.

Proof of (b): Let subtask T_k be T_i 's successor. By (3) and (6), $r(T_k) \geq d(T_i) - b(T_i)$. (Recall that consecutive PF-windows overlap by at most one slot.) If $d(T_i) > t + 1$ or $d(T_i) = t + 1 \wedge b(T_i) = 0$, then $r(T_k) \geq t + 1$. Since T_i is scheduled in slot t , T_k is scheduled at or after $t + 1$. Therefore, by (a), $e(T_k) \geq t + 1$.

Proof of (c): Suppose τ contains a subtask U_j with a deadline greater than t_d . U_j can be removed without affecting the scheduling of higher-priority subtasks with earlier deadlines. Thus, if U_j is removed, then a deadline is still missed at t_d . This contradicts (T2).

Proof of (d): If there were a hole in slot $t_d - 1$, then the subtask that misses its deadline at t_d would have been scheduled there, a contradiction. (Note that its predecessor meets its deadline at or before time $t_d - 1$ and hence is not scheduled in slot $t_d - 1$.)

Proof of (e): By (10), we have

$$LAG(\tau, t_d) = \sum_{t=0}^{t_d-1} \sum_{T \in \tau} flow(T, t) - \sum_{t=0}^{t_d-1} \sum_{T \in \tau} S(T, t).$$

The first term on the right-hand side of the above equation is the total share in $[0, t_d)$, which equals the total number of subtasks in τ . The second term corresponds to the number of subtasks scheduled by EPDF in $[0, t_d)$. Since exactly one subtask misses its deadline, the difference between these two terms is 1, *i.e.*, $LAG(\tau, t_d) = 1$.

Proof of (f): By (d), there are no holes in slot $t_d - 1$. Hence, by Lemma 4, $LAG(\tau, t_d - 1) \geq LAG(\tau, t_d)$. Therefore, by (e), $LAG(\tau, t_d - 1) \geq 1$. \square

Because $LAG(\tau, 0) = 0$, by part (f) of Lemma 9, there exists a time t such that

$$0 \leq t < t_d - 1 \wedge LAG(\tau, t) < 1 \wedge LAG(\tau, t + 1) \geq 1. \quad (13)$$

Without loss of generality, let t be the latest such time, *i.e.*, for all u such that $t < u \leq t_d - 1$, $LAG(\tau, u) \geq 1$. We now show that such a t cannot exist, thus contradicting our starting assumption that t_d and τ exist.

By (13), $LAG(\tau, t) < LAG(\tau, t + 1)$. Hence, by Lemma 4, *there is at least one hole in slot t* . Define sets A , B , and I as in Sec. 4 (refer to Fig. 5). We begin by proving certain properties about B .

Lemma 10 *B is non-empty.*

Proof: Let the number of the holes in slot t be h . Then, $\sum_{T \in \tau} S(T, t) = M - h$. By (10), $LAG(\tau, t + 1) = LAG(\tau, t) + \sum_{T \in \tau} (flow(T, t) - S(T, t))$. Thus, because $LAG(\tau, t) < LAG(\tau, t + 1)$, we have $\sum_{T \in \tau} flow(T, t) > M - h$. Since for every $V \in I$, either $d(V_k) < t$ or $r(V_k) > t$, by (9), $flow(V, t) = 0$. It follows that $\sum_{T \in A \cup B} flow(T, t) > M - h$. Therefore, by (F1), $\sum_{T \in A \cup B} wt(T) > M - h$. Because the number of tasks scheduled in slot t is $M - h$, $|A| = M - h$. Because $wt(T) \leq 1$ for any task T , $\sum_{T \in A} wt(T) \leq M - h$. Thus, $\sum_{T \in B} wt(T) > 0$. Hence, B is not empty. \square

Lemmas 11–13, and 15 below are proved in an appendix. In the proof of each of Lemmas 11–13, we show that if the required condition is not satisfied, then a subtask can be removed without causing the missed deadline at t_d to be met. Thus, we obtain a contradiction of (T2).

Lemma 11 *Let U be any task in B . Let U_j be the subtask with the largest index such that $e(U_j) \leq t < d(U_j)$. Then, $d(U_j) = t + 1 \wedge b(U_j) = 1$.*

Lemma 12 *If B has at least one light task, then there is no hole in slot $t + 1$.*

Lemma 13 *Let U be a heavy task in B and let U_j be the subtask of U with the largest index such that $e(U_j) \leq t < d(U_j)$. Then, there exists a slot with no holes in $[d(U_j), \min(D(U_j), t_d))$.*

Lemma 14 *If B has at least one light task, then $LAG(\tau, t + 2) < 1$.*

Proof: Let the number of holes in slot t be h . We now derive some properties about the *flow* values in slots t and $t + 1$.

By the definition of I , only tasks in $A \cup B$ are active at time t . Thus, $\sum_{T \in \tau} flow(T, t) = \sum_{T \in A \cup B} flow(T, t)$. Since $wt(T) \leq 1$ for any T ,

we have $\sum_{T \in A} wt(T) \leq |A|$. Thus, by (F1), $\sum_{T \in A} flow(T, t) \leq |A|$. Now, because there are h holes in slot t , $M - h$ tasks are scheduled at t , *i.e.*, $|A| = M - h$. Thus, $\sum_{T \in A} flow(T, t) \leq M - h$ and

$$\sum_{T \in \tau} flow(T, t) \leq M - h + \sum_{T \in B} flow(T, t). \quad (14)$$

Consider $U \in B$. Let U_j be the subtask of U with the largest index such that $e(U_j) \leq t < d(U_j)$. Let C denote the set of such subtasks for all tasks in B . Then, by Lemma 11,

$$\text{for all } U_j \in C, d(U_j) = t + 1 \wedge b(U_j) = 1. \quad (15)$$

If U is heavy, then this would imply that $D(U_j) > t + 1$. (By the definition of a group deadline, for any subtask T_i of a heavy task T , $D(T_i) = d(T_i)$ holds iff $b(T_i) = 0$.) Thus, the leave condition in (C2) is not satisfied at time $t + 1$, and hence no task in B leaves at time $t + 1$.

Let A' (I') denote the tasks in A (I) that are active at time $t + 1$. Then, the set of active tasks at time $t + 1$ is $A' \cup I' \cup B$. Thus, by the join condition in (C2),

$$\sum_{T \in A' \cup I' \cup B} wt(T) \leq M. \quad (16)$$

Also, $\sum_{T \in \tau} flow(T, t + 1) = \sum_{T \in A' \cup I' \cup B} flow(T, t + 1)$. By (F1), this implies that $\sum_{T \in \tau} flow(T, t + 1) \leq \sum_{T \in A' \cup I'} wt(T) + \sum_{T \in B} flow(T, t + 1)$. Thus, by (14),

$$\begin{aligned} \sum_{T \in \tau} (flow(T, t) + flow(T, t + 1)) &\leq M - h + \\ \sum_{T \in A' \cup I'} wt(T) + \sum_{T \in B} (flow(T, t) + flow(T, t + 1)) & \end{aligned} \quad (17)$$

Consider $U_j \in C$ (hence, $U \in B$). Let U_k denote the successor of U_j . Since U_j is the subtask with the largest index such that $e(U_j) \leq t < d(U_j)$, we have $e(U_k) \geq t + 1$. Hence, $r(U_k) \geq t + 1$. By (15), we have $d(U_j) = t + 1$. Therefore, by (F2), $flow(U, t) + flow(U, t + 1) \leq wt(U)$ for each $U \in B$. By (17), this implies that $\sum_{T \in \tau} (flow(T, t) + flow(T, t + 1)) \leq M - h + \sum_{T \in A' \cup I' \cup B} wt(T)$. Thus,

from (16), it follows that

$$\sum_{T \in \tau} (\text{flow}(T, t) + \text{flow}(T, t+1)) \leq M - h + M. \quad (18)$$

By the statement of the lemma, B contains at least one light task. Therefore, by Lemma 12, there is no hole in slot $t+1$. Since there are h holes in slot t , we have $\sum_{T \in \tau} (S(T, t) + S(T, t+1)) = M - h + M$.

Hence, by (18), $\sum_{T \in \tau} (\text{flow}(T, t) + \text{flow}(T, t+1)) \leq \sum_{T \in \tau} (S(T, t) + S(T, t+1))$. Using this relation in the identity (obtained from (10)), $LAG(\tau, t+2) = LAG(\tau, t) + \sum_{T \in \tau} (\text{flow}(T, t) + \text{flow}(T, t+1)) - \sum_{T \in \tau} (S(T, t) + S(T, t+1))$, and the fact that $LAG(\tau, t) < 1$, we obtain $LAG(\tau, t+2) < 1$. \square

The following lemma generalizes Lemma 14 by allowing B to consist solely of heavy tasks.

Lemma 15 *There exists $v \in \{t+2, \dots, t_d\}$ such that $LAG(\tau, v) < 1$.*

Recall our assumption that t is the latest time such that $LAG(\tau, t) < 1$ and $LAG(\tau, t+1) \geq 1$. Because $t \leq t_d - 2$ (by (13)), we have $t+2 \leq t_d$. By Lemma 15, $LAG(\tau, v) \leq 0$ for some $v \in \{t+2, \dots, t_d\}$. By Lemma 9, parts (e) and (f), v cannot be t_d or $t_d - 1$. Thus, $v \leq t_d - 2$. Because $LAG(\tau, t_d) \geq 1$, this contradicts the maximality of t . Therefore, t_d and τ as defined cannot exist. Thus, we have the following.

Theorem 4 *PD^2 correctly schedules any dynamic GIS task system satisfying (C2).*

6 Conclusions

In this paper, we have addressed the problem of scheduling dynamic GIS task systems on multiprocessors. We have shown that if the sum of the weights of the $M - 1$ heaviest tasks is at most 1, then the uniprocessor join/leave conditions presented previously [5, 11] are sufficient to avoid deadline misses on M processors when EDPF is used. This result applies to any EPDF-based algorithm, and hence to PD^2 as well. We have also provided join/leave conditions for the general case in which weights are not restricted in this way and

tasks are scheduled using PD^2 . We have further shown that, in general, it is not possible to improve upon these conditions.

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Appendix: Remaining Proofs

In this appendix, we present all proofs omitted earlier. In all the lemmas, the time t corresponds to that defined in Equation (13).

Lemma 11 *Let U be any task in B . Let U_j be the subtask with the largest index such that $e(U_j) \leq t < d(U_j)$. Then, $d(U_j) = t + 1 \wedge b(U_j) = 1$.*

Proof: As shown in Sec. 4, U_j must be scheduled before t . By (13), $t < t_d$. Hence, U_j does not miss its deadline and $d(U_j) \geq t + 1$. Suppose that the following holds.

$$d(U_j) > t + 1 \text{ or } d(U_j) = t + 1 \wedge b(U_j) = 0 \quad (19)$$

We now show that U_j can be removed and a deadline will still be missed at t_d , contradicting (T2). Let the chain of displacements caused by removing U_j be $\Delta_1, \Delta_2, \dots, \Delta_k$, where $\Delta_i = \langle X^{(i)}, t_i, X^{(i+1)}, t_{i+1} \rangle$ and $X^{(1)} = U_j$. By Lemma 1, $t_{i+1} > t_i$ for $1 \leq i \leq k$.

Note that at slot t_i , the priority of subtask $X^{(i)}$ is at least that of $X^{(i+1)}$, because $X^{(i)}$ was chosen over $X^{(i+1)}$ in S . Thus, because $X^{(1)} = U_j$, by (19), for each subtask $X^{(i)}$, $1 \leq i \leq k + 1$, either $d(X^{(i)}) > t + 1$ or $d(X^{(i)}) = t + 1 \wedge b(X^{(i)}) = 0$. Therefore, by part (b) of Lemma 9, the following property holds.

(E) The eligibility time of the successor of $X^{(i)}$ (if it exists in τ) is at least $t + 1$ for all $i \in [1, k + 1]$.

We now show that the displacements do not extend beyond slot t . Assume, to the contrary, that $t_{k+1} > t$. Consider $h \in \{2, \dots, k + 1\}$ such that $t_h > t$ and $t_{h-1} \leq t$, as depicted in Fig. 6(a). Such an h exists because $t_1 < t < t_{k+1}$. Because there is a hole in slot t and $t_{h-1} \leq t < t_h$, by Lemma 3, $t_{h-1} = t$ and $X^{(h)}$ must be $X^{(h-1)}$'s successor. Therefore, by (E), $e(X^{(h)}) \geq t + 1$. This implies that Δ_{h-1} is not valid.

Thus, the displacements do not extend beyond slot t , implying that no subtask scheduled after t is left-shifted. Hence, a deadline is still missed at time t_d , contradicting (T2). Hence, $d(U_j) = t + 1 \wedge b(U_j) = 1$. \square

The following property is used in the proof of the next lemma. It follows from the fact that light tasks have PF-windows of length at least three and consecutive PF-windows overlap by at most one slot [1].

(L) For a light task T , if T_k is the successor of T_i , then $d(T_k) \geq d(T_i) + 2$.

Lemma 12 *If B has at least one light task, then there is no hole in slot $t + 1$.*

Proof: By (13), $t < t_d - 1$, and therefore, $t + 1 \leq t_d - 1$. Suppose that there is a hole in slot $t + 1$. By part (d) of Lemma 9, $t + 1 < t_d - 1$, *i.e.*,

$$t + 2 \leq t_d - 1. \quad (20)$$

Let U be a light task in B and let U_j be the subtask of U with the largest index such that $e(U_j) \leq t < d(U_j)$. Our approach is the same as in the proof of Lemma 11. Let the chain of displacements caused by removing U_j be $\Delta_1, \Delta_2, \dots, \Delta_k$, where $\Delta_i = \langle X^{(i)}, t_i, X^{(i+1)}, t_{i+1} \rangle$ and $X^{(1)} = U_j$. By Lemma 1, we have $t_{i+1} > t_i$ for all $i \in [1, k]$. Also, the priority of $X^{(i)}$ is at least that of $X^{(i+1)}$ at t_i , because $X^{(i)}$ was chosen over $X^{(i+1)}$ in S . Because U is light and $d(U_j) = t + 1 \wedge b(U_j) = 1$ (by Lemma 11), this implies the following.

(P) For all $i \in [1, k + 1]$, either **(i)** $d(X^{(i)}) > t + 1$ or **(ii)** $d(X^{(i)}) = t + 1$ and $X^{(i)}$ is the subtask of a light task.

Suppose the chain of displacements extends beyond $t + 1$, *i.e.*, $t_{k+1} > t + 1$. Consider $h \in \{1, \dots, k + 1\}$ such that that $t_h > t + 1$ and $t_{h-1} \leq t + 1$. Because there is a hole in slot $t + 1$ and $t_{h-1} \leq t + 1 < t_h$, by Lemma 3, $t_{h-1} = t + 1$ and $X^{(h)}$ is the successor of $X^{(h-1)}$. Similarly, because there is a hole in slot t , $t_{h-2} = t$ and $X^{(h-1)}$ is the successor of $X^{(h-2)}$. This is illustrated in Fig. 6(b).

By (P), either $d(X^{(h-2)}) > t + 1$ or $d(X^{(h-2)}) = t + 1$ and $X^{(h-2)}$ is the subtask of a light task. In either case, $d(X^{(h-1)}) > t + 2$. To see why, note that if $d(X^{(h-2)}) > t + 1$, then because $X^{(h-1)}$ is the successor of $X^{(h-2)}$, by (5), $d(X^{(h-1)}) > t + 2$. On the other hand, if $d(X^{(h-2)}) = t + 1$ and $X^{(h-2)}$ is the subtask of a light task, then, by (L), $d(X^{(h-1)}) > t + 2$.

Now, because $X^{(h-1)}$ is scheduled at $t + 1$, by part (b) of Lemma 9, the successor of $X^{(h-1)}$ is not eligible before $t + 2$, *i.e.*, $e(X^{(h)}) \geq t + 2$. This implies that the displacement Δ_{h-1} is not valid. Thus, the chain of displacements cannot extend beyond time $t + 2$. Hence, because $t + 2 \leq t_d - 1$ (by (20)), removing U_j cannot cause a missed deadline at t_d to be met. This contradicts (T2). Hence, there is no hole in slot $t + 1$. \square

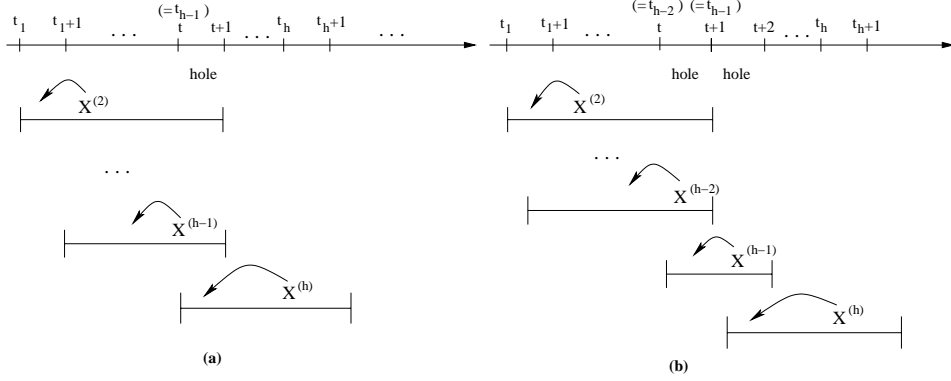


Figure 6: IS-windows are denoted by line segments. **(a)** Lemma 11. $X^{(h)}$ must be the successor of $X^{(h-1)}$ because there is a hole in slot t . **(b)** Lemma 12. If there is a hole in both slots t and $t+1$, then $X^{(h-2)}$ and $X^{(h-1)}$ must be scheduled at t and $t+1$ in S , respectively. Also, in τ , $X^{(h)}$ must be the successor of $X^{(h-1)}$, which in turn, must be the successor of $X^{(h-2)}$.

The following lemma is the counterpart of Lemma 12 for heavy tasks.

Lemma 13 *Let U be a heavy task in B and let U_j be the subtask of U with the largest index such that $e(U_j) \leq t < d(U_j)$. Then, there exists a slot with no holes in $[d(U_j), \min(D(U_j), t_d)]$.*

Proof: By Lemma 11, $d(U_j) = t+1 \wedge b(U_j) = 1$. By (13), $t < t_d - 1$. Therefore $d(U_j) \leq t_d - 1$. If $\min(D(U_j), t_d) = t_d$, then by part (f) of Lemma 9, slot $t_d - 1$ satisfies the stated requirement. In the rest of the proof, assume that $D(U_j) < t_d$. Let $v = D(U_j)$. Since $b(U_j) = 1$, by the definition of D , $D(U_j) > d(U_j)$, i.e.,

$$t+1 < v. \quad (21)$$

Suppose that the following property holds.

(H) There is a hole in slot u for all $u \in \{t, \dots, v-1\}$.

Given (H), we show that removing U_j does not cause the missed deadline to be met, contradicting (T2). Let $\Delta_1, \Delta_2, \dots, \Delta_k$ be the chain of displacements caused by removing U_j , where $\Delta_i = \langle X^{(i)}, t_i, X^{(i+1)}, t_{i+1} \rangle$, $X^{(1)} = U_j$, and t_1 is the slot in which U_j is scheduled. By Lemma 1, $t_{i+1} > t_i$ for all $i \in [1, k-1]$. Also, the priority of $X^{(i)}$ is at least that of $X^{(i+1)}$ at t_i , because $X^{(i)}$ was chosen over $X^{(i+1)}$ at t_i in S . Thus, by Lemma 11, for all $i \in [2, k+1]$, one of the following holds:

- (a)** $d(X^{(i)}) > t+1$,
- (b)** $d(X^{(i)}) = t+1 \wedge b(X^{(i)}) = 0$, or
- (c)** $d(X^{(i)}) = t+1 \wedge b(X^{(i)}) = 1 \wedge D(X^{(i)}) \leq v$.

We now show that the displacements do not extend beyond slot $v-1$ (which implies that U_j can be removed without causing the missed deadline to be met).

Suppose, to the contrary, they do extend beyond slot $v-1$, i.e., $t_{k+1} > v-1$.

Let t_g be the largest t_i such that $t_i < t$ and let t_h be the smallest t_i such that $t_i > v-1$. (Note that such a t_g exists because $t_1 < t$.) Then, by (H), there are holes in all slots in $[t_{g+1}, t_{h-1}]$. Thus, by Lemma 3,

$$\forall i \in [g+1, h-1], X^{(i+1)} \text{ is the successor of } X^{(i)}. \quad (22)$$

Also, $t_{i+1} = t_i + 1$ for all $i \in [g+1, h-2]$.

$$t_{g+1} = t \wedge t_{h-1} = v-1 \quad (23)$$

$$\forall i \in \{g+1, \dots, h-1\}, t_i = t + i - (g+1) \quad (24)$$

Earlier, we showed that one of (a)–(c) holds for all $i \in [2, k+1]$. If either $d(X^{(g+1)}) > t+1$ or $d(X^{(g+1)}) = t+1 \wedge b(X^{(g+1)}) = 0$, then since $X^{(g+1)}$ is scheduled at t , by Lemma 9, part (b), $e(X^{(g+2)}) \geq t+1$ (recall that, by (22), $X^{(g+2)}$ is the successor of $X^{(g+1)}$). In other words, the displacement Δ_g is not valid. Therefore,

$$d(X^{(g+1)}) = t+1 \wedge b(X^{(g+1)}) = 1 \wedge D(X^{(g+1)}) \leq v. \quad (25)$$

We now consider two cases. In each, we show that the displacements do not extend beyond $v-1$, as desired.

Case 1: $X^{(g+1)}$ is the subtask of a light task.

By (21), $t+1 \leq v-1$ and hence, by (H), there is a hole in both t and $t+1$. Also, by (23) and (24), we have $v-1 = t + (h-1) - (g+1) = t + h - g - 2$. Because $t < v-1$ (by (21)), we have $h > g+2$, i.e.,

$$h \geq g+3.$$

Because $X^{(g+1)}$ is the subtask of a light task, the reasoning used in the proof of Lemma 12 applies. Thus,

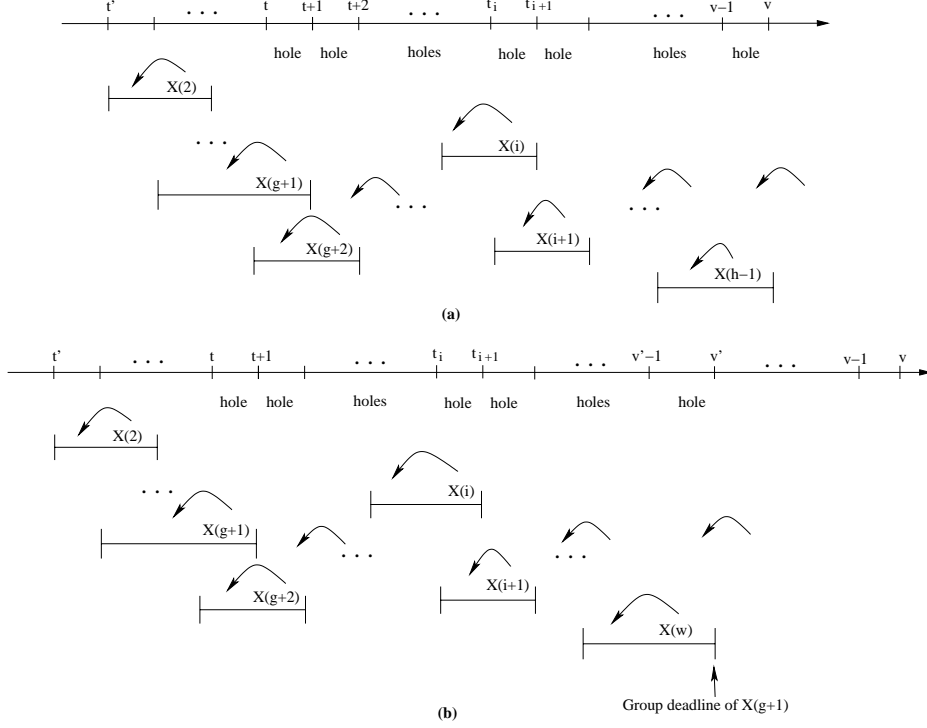


Figure 7: Lemma 13. (a) There are holes in all slots in $[t, v)$. $X^{(i)}$ scheduled at t_i displaces $X^{(i-1)}$ scheduled at t_{i-1} . By (24), the t_i 's are consecutive and satisfy $t_i = t + i - (g + 1)$. Further, $X^{(h-1)}$ is the subtask scheduled in slot $v - 1$. (b) Case 2. $D(X^{(g+1)}) = v'$. Hence, either $d(X^{(w)}) = v' \wedge b(X^{(w)}) = 0$ (as depicted) or $d(X^{(w)}) > v'$.

the displacement Δ_{g+2} is not valid. Hence, the displacements do not extend beyond slot $t + 1$ (and hence, slot $v - 1$).

Case 2: $X^{(g+1)}$ is the subtask of a heavy task.

Let $v' = D(X^{(g+1)})$. By (25), $v' \leq v$. We now show that the displacements cannot extend beyond slot $v' - 1$ (and hence, slot $v - 1$). By (24), $X^{(i)}$ is scheduled in slot $t + i - (g + 1)$ in S for all $i \in \{g + 1, \dots, h - 1\}$. By (22), all $X^{(i)}$ where $g + 1 \leq i \leq h$ are subtasks of the same heavy task. We now show that the displacement $\Delta_{v'-1-t+(g+1)} (= \Delta_{v'-t+g})$ is not valid. Let $w = v' - t + g$.

By (24), $t_w = v' - 1$. Because $X^{(i)}$ is scheduled at t_i , the subtask scheduled at $v' - 1$ is $X^{(w)}$. Since $X^{(i+1)}$ is the successor of $X^{(i)}$, by (5), $d(X^{(i)}) > d(X^{(i-1)})$ for all $i \in [g + 2, w]$. Because $d(X^{(g+1)}) = t + 1$ (by (25)),

$$\forall i \in [g + 1, w], d(X^{(i)}) \geq t + i - g. \quad (26)$$

In particular, $d(X^{(w)}) \geq v'$.

We now show that if $d(X^{(w)}) = v'$, then $b(X^{(w)}) = 0$. In this case, because $d(X^{(w-1)}) < d(X^{(w)})$, we have $d(X^{(w-1)}) < v'$. By (26), $d(X^{(w-1)}) \geq v' - 1$. Therefore, $d(X^{(w-1)}) = v' - 1$. Similarly, by induction, $d(X^{(i)}) = v' - 1 + i - g$ for all $i \in [g + 1, w]$. (Refer to

Fig. 7(b).) Because $D(X^{(g+1)}) = v'$, by the definition of D , $b(X^{(v'-u+g+1)}) = 0$. (In this case, the group deadline corresponds to the last slot of a window of length two.)

Thus, either $d(X^{(w)}) > v'$ or $d(X^{(w)}) = v' \wedge b(X^{(w)}) = 0$. Since $X^{(w)}$ is scheduled at $v' - 1$, by Lemma 9, part (b), the eligibility time of the successor of $X^{(w)}$ is at least v' . Hence, Δ_w is not valid. Thus, the displacements do not extend beyond slot $v' - 1$. \square

The following claims are used in proving Lemma 15.

Claim 1 *If U_j is scheduled in slot u , where $0 \leq u < t_d$ and $u \leq d(U_j)$, and if there is a hole in slot u , then $d(U_j) = u + 1$.*

Proof: Because $u < t_d$, by Def. 1, no deadline is missed in $[0, u + 1)$. Because U_j is scheduled in slot u , i.e., $[u, u + 1)$, we have $d(U_j) \geq u + 1$. Suppose that $d(U_j) > u + 1$. Then, by part (b) of Lemma 9, the successor of U_j (if it exists) is not eligible before $u + 1$. Hence, by Lemma 2, we can remove U_j and no displacements will result, i.e., a deadline is still missed at t_d , contradicting (T2). Therefore, $d(U_j) = u + 1$. \square

Claim 2 *Suppose there is a hole in slot $u \in \{0, \dots, t_d - 1\}$. Let U_j be a subtask scheduled at $t' \leq u$. If the*

eligibility time of the successor of U_j is at least $u + 1$, then $d(U_j) \leq u + 1$.

Proof: If $t' = u$, then by Claim 1, $d(U_j) = u + 1$. On the other hand, if $t' < u$ and $d(U_j) \geq u$, then by Lemma 11, $d(U_j) = u + 1$. \square

We use the following property about flows (proved in [10]) in the proof of Lemma 15.

(F3) Let T_i be a subtask of a heavy task T such that $b(T_i) = 1$ and let T_k be the successor of T_i . If $u \in \{d(T_i), \dots, D(T_i) - 1\}$ and $u \leq r(T_k)$, then $flow(T, d(T_i)) + flow(T, u) \leq wt(T)$.

Lemma 15 *There exists $v \in \{t + 2, \dots, t_d\}$ such that $LAG(\tau, v) < 1$.*

Proof: Because $LAG(\tau, t) < 1$ and $LAG(\tau, t + 1) \geq 1$ (by (13)),

$$LAG(\tau, t) < LAG(\tau, t + 1). \quad (27)$$

Thus, by Lemma 4, we have the following property.

(H) There is at least one hole in slot t .

Let A, B , and I be as defined in the proof of Lemma 14. If any task in B is light, then by Lemma 14, $LAG(\tau, t + 2) \leq 0$, which establishes our proof obligation. We henceforth assume all tasks in B are heavy.

Let U be any task in B . Let U_j be the subtask with the largest index such that $e(U_j) \leq t < d(U_j)$. Let C denote the set of such subtasks of all tasks in B . Then, by Lemma 11,

$$\forall U_j \in C, d(U_j) = t + 1 \wedge b(U_j) = 1. \quad (28)$$

Let L_i be the lowest-priority subtask in C . Then,

$$\forall U_j \in C, d(U_j) = t + 1 \wedge b(U_j) = 1 \wedge D(U_j) \geq D(L_i). \quad (29)$$

By Lemma 13, there is a slot in $[t, \min(D(L_i), t_d)]$ with no hole. Let u be as follows.

(U) u is the earliest slot in $[t, \min(D(L_i), t_d)]$ with no hole.

Fig. 8 depicts this situation. By (U) and (H),

$$u \geq t + 1, \quad (30)$$

and there are holes in all slots in $\{t, \dots, u - 1\}$. We now establish the following property about tasks in B .

Claim 3 *All tasks in B are inactive over the interval $[t + 1, u)$.*

Proof: If the interval $[t + 1, u)$ is empty, then the claim is vacuously true, so assume it is nonempty.

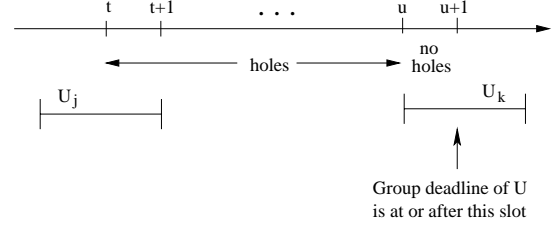


Figure 8: Lemma 15. $U_j \in C$ and U_k is the successor of U_j . There is a hole in each slot in $[t, u)$ and there is no hole in slot u . The earliest time at which U_k 's PF-window starts is u , i.e., $r(U_k) \geq u$.

Let V be any task in B . We first show that no subtask of V is scheduled in $[t, u)$.

Note that because $V \in B$, no subtask of V is scheduled in slot t . Let V_i be the earliest subtask of V scheduled in $[t + 1, u)$ and let v be the slot in which it is scheduled. Because there is hole in slot v , by Claim 1, $d(V_i) = v + 1$. By (4) and (5), this implies that $r(V_i) \leq v$. If $r(V_i) < v$, then $e(V_i) < v$. Thus, because there are holes in all slots in $\{t, \dots, v - 1\}$, it should have been scheduled earlier. Therefore, $r(V_i) = v$, which implies that $wt(V) = 1$. However, this contradicts the fact that some subtask of V has a b -bit of 1 (by (28)). Hence, no subtask of any task in B is scheduled in $[t, u)$ (see Fig. 8). Moreover, because there are holes in all slots in $[t, u)$, the earliest slot after t at which a subtask of a task in B is eligible to be scheduled is u . By (28), this implies that all the tasks in B are inactive in $[t + 1, u - 1]$. \square

For any $U_j \in C$, by (29) and (U), $D(U_j) > u$. Therefore, by (C2), task U cannot leave before time $u + 1$. Thus, no task in B can leave before time $u + 1$.

Let U_j be any subtask in C , and let U_k be the successor of U_j . By Claim 3, $r(U_k) \geq u$. Furthermore, by (28)–(30) and (U), $d(U_j) = t + 1 \leq u < D(U_j)$. Hence, by (F3), $flow(U, t) + flow(U, u) \leq wt(U)$. Because this argument applies to all tasks in B , we have

$$\forall U \in B, flow(U, t) + flow(U, u) \leq wt(U). \quad (31)$$

We now show that LAG is non-increasing over $[t + 1, u)$.

Claim 4 $LAG(\tau, v + 1) \leq LAG(\tau, v)$ for all $v \in \{t + 1, \dots, u - 1\}$,

Proof: If $\{t + 1, \dots, u\}$ is empty, then the claim is vacuously true, so assume it is nonempty. Suppose for some $v \in \{t + 1, \dots, u - 1\}$, $LAG(\tau, v + 1) > LAG(\tau, v)$. Then, by Lemma 10, there exists a task that is active at v but not scheduled at v . Let V be one such task and let V_k be the subtask

with the largest index such that

$$e(V_k) \leq v < d(V_k). \quad (32)$$

Because no subtask of V is scheduled at v and because there is a hole at v , V_k is scheduled before v . By (U), there is a hole at $v - 1$; moreover, because $t + 1 \leq v \leq u - 1$, we have $v - 1 \in \{t, \dots, u - 2\} \subseteq \{0, \dots, t_d - 1\}$. Hence, by Claim 2, we have $d(V_k) \leq v$, which contradicts (32). Therefore, $LAG(\tau, v + 1) \leq LAG(\tau, v)$ for all $v \in [t + 1, u - 1]$. \square

We now show that $LAG(\tau, u + 1) \leq 0$, which establishes our proof obligation.

For each $v \in \{t, \dots, u\}$, let H_v denote the number of holes in slot v . Then, $M - H_v$ tasks are scheduled in slot v . Also, let I_v (A_v) denote the tasks in I (A) that are active at v .

By (10) and Claim 4, $\sum_{T \in \tau} flow(T, v) \leq \sum_{T \in \tau} S(T, v)$. Therefore,

$$\forall v \in \{t + 1, \dots, u - 1\}, \sum_{T \in \tau} flow(T, v) \leq M - H_v. \quad (33)$$

By the join condition in (C2) and by (7), we have $\sum_{T \in B \cup I_u \cup A_u} wt(T) \leq M$. Hence, by (31) and (F1), we get $\sum_{T \in B} (flow(T, t) + flow(T, u)) + \sum_{T \in I_u \cup A_u} flow(T, u) \leq M$. Thus,

$$\sum_{T \in B} flow(T, t) + \sum_{T \in B \cup I_u \cup A_u} flow(T, u) \leq M. \quad (34)$$

Because the tasks in A ($= A_t$) are the ones scheduled in slot t , the number of tasks in set A_t is $M - H_t$. Hence, by (F1) and because the weight of each task is at most one,

$$\sum_{T \in A_t} flow(T, t) \leq \sum_{T \in A_t} wt(T) \leq M - H_t. \quad (35)$$

We are now ready to show that $LAG(\tau, u + 1) \leq 0$. Because $S(T, v) = M - H_v$, by (10), $LAG(\tau, u + 1) - LAG(\tau, t) = R$, where $R = \sum_{v=t}^u (\sum_{T \in \tau} flow(T, v)) - \sum_{v=t}^u (M - H_v)$. Because, by (U), there are no holes in slot u , $H_u = 0$. Therefore,

$$R = \sum_{v=t}^u \left(\sum_{T \in \tau} flow(T, v) \right) - \sum_{v=t}^{u-1} (M - H_v) - M. \quad (36)$$

The right-hand side of (36) can be rewritten as follows.

$$\sum_{T \in \tau} (flow(T, t) + flow(T, u)) - (M - H_t) - M$$

$$+ \sum_{v=t+1}^{u-1} \left(\sum_{T \in \tau} flow(T, v) - (M - H_v) \right).$$

Rearranging terms, and using $\sum_{T \in I} flow(T, t) = 0$ (which follows by the definition of I), we get

$$\begin{aligned} & \sum_{T \in B} flow(T, t) + \sum_{T \in B \cup I_u \cup A_u} flow(T, u) - M \\ & + \sum_{T \in A_t} flow(T, t) - (M - H_t) \\ & + \sum_{v=t+1}^{u-1} \left(\sum_{T \in \tau} flow(T, v) - (M - H_v) \right). \end{aligned}$$

By (33)–(35), the above value is non-positive. Hence, by (36), $LAG(\tau, u + 1) - LAG(\tau, t) \leq 0$. Because $LAG(\tau, t) < 1$, this implies that $LAG(\tau, u + 1) < 1$.

By (U) and (30), $t + 1 \leq u \leq \min(D(U_j), t_d) - 1$. Hence, $t + 2 \leq u + 1 \leq t_d$. Thus, there exists a $v \in \{t + 2, \dots, t_d\}$ such that $LAG(\tau, v) < 1$. \square